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ASYMPTOTIC RATIO OF HARMONIC MEASURES OF SLIT SIDES

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The article is devoted to the geometry of solutions to the chordal Löwner equation which is based on the comparison of singular solutions and harmonic measures for the sides of a slit in the upper half-plane generated by a driving term. An asymptotic ratio for harmonic measures of slit sides is found for a slit which is tangential to a straight line under a given angle, and for a slit with high order tangency to a circular arc tangential to the real axis.

Key words: Löwner equation, singular solution, harmonic measure, half-plane capacity.

INTRODUCTION

The famous Löwner differential equation have been introduced in 1923 [1] and was aimed to give a parametric representation of slit domains. In this article we describe an asymptotic behavior of singular solutions and harmonic measures for the sides of a slit in domains generated by a driving term of the Löwner equation.

The chordal version of the Löwner equation deals with the upper half-plane $\mathbb{H} = \{z : \text{Im } z > 0\}$, $\mathbb{R} = \partial\mathbb{H}$, and functions $f(z, t)$ normalized near infinity by

$$f(z, t) = z + \frac{2t}{z} + O\left(\frac{1}{z^2}\right)$$

which solve the chordal Löwner differential equation

$$\frac{df(z, t)}{dt} = \frac{2}{f(z, t) - \lambda(t)}, \quad f(z, 0) \equiv z, \quad t \geq 0, \quad (1)$$

and map subdomains of \mathbb{H} onto \mathbb{H} . Here $\lambda(t)$ is a real-valued continuous driving term.

Let $\gamma_t := \gamma[0, t] = \{\gamma(x) : 0 \leq x \leq t\}$ be a simple continuous curve in $\mathbb{H} \cup \{0\}$ with endpoints $\gamma(0) = 0$ and $\gamma(t)$, $0 \leq t \leq T$. Then there is a unique map $f(z, t) : \mathbb{H} \setminus \gamma_t \rightarrow \mathbb{H}$ satisfying the chordal Löwner equation (1) with $\lambda(t)$ uniquely determined by $\gamma[0, t]$. The function $f(z, t)$ can be extended continuously to $\mathbb{R} \cup \gamma(t)$, and $f(\gamma(t), t) = \lambda(t)$. The value t is called the *half-plane capacity* of the curve γ_t , $t = \text{hcap}(\gamma_t)$, see, e.g. [2].

We say that $\gamma_t \in C^n$, $n \in \mathbb{N}$, on $[0, S]$ if, for the arc-length parameter s of γ_t , $\gamma(t(s))$ has a continuous derivative $\gamma^{(n)}$ in s on $[0, S]$, $t(S) = T$. All the derivatives $\gamma^{(k)}$, $1 \leq k \leq n$, at $s = 0$ are understood as one-side derivatives. A curve $\gamma_t \in C^n$, $\gamma(0) = 0$, is said to have at least n -order tangency with a ray $I_\theta = \{e^{i\theta}s : s \geq 0\}$, $\theta \in \mathbb{R}$, at $s = 0$ if

$$\gamma(t(s)) = e^{i\theta}s + o(s^n), \quad s \rightarrow +0.$$



Two curves $\gamma[0, s] \in C^n$ and $\Gamma[0, s] \in C^n$ are said to have at least n -order tangency at $s = 0$ if derivatives $\gamma^{(k)}$ and $\Gamma^{(k)}$ in s at $s = 0$ coincide, $0 \leq k \leq n$.

The extended function $f(z, t)$ maps γ_t onto a segment $I = I(t) = [f_2(0, t), f_1(0, t)]$ while $f(\mathbb{R}) = \mathbb{R} \setminus I$. The function $f_1(0, t)$ is the maximal singular solution to the chordal Löwner equation (1), and $f_2(0, t)$ is the minimal singular solution to (1). Both of these solutions correspond to the singular point $f(0, 0) = 0$ of equation (1).

The curve γ_t has two sides γ_{1t} and γ_{2t} which define different prime ends at the same points, except for its tip. We say that γ_{1t} is the *left* side of γ_t if going along the boundary of the domain $\mathbb{H} \setminus \gamma_t$ and moving along \mathbb{R} from $(-\infty)$ to 0 , we first meet the side γ_{1t} and then γ_{2t} . In this case, γ_{2t} is called the *right* side of γ_t . The two parts $[f_2(0, t), \lambda(t)]$ and $[\lambda(t), f_1(0, t)]$ of segment $I(t)$ are the images of the two sides γ_{1t} and γ_{2t} of γ_t under $f(z, t)$, respectively.

The harmonic measures $\omega(f^{-1}(i, t); \gamma_{kt}, \mathbb{H} \setminus \gamma_t)$ of γ_{kt} at $f^{-1}(i, t)$ with respect to $\mathbb{H} \setminus \gamma_t$ are defined by the functions ω_k which are harmonic on $\mathbb{H} \setminus \gamma_t$ and continuously extended on its closure except for the endpoints of γ_t , $\omega_k|_{\gamma_{kt}} = 1$, $\omega_k|_{\mathbb{R} \cup (\gamma_t \setminus \gamma_{kt})} = 0$, $k = 1, 2$, see, e.g., [3, § 3.6]. Denote

$$m_k(t) := \omega(f^{-1}(i, t); \gamma_{kt}, \mathbb{H} \setminus \gamma_t), \quad k = 1, 2.$$

In Section 1, we prove the following theorem.

Theorem 1. *Let $\gamma_t \in C^4$, $\gamma(0) = 0$, $\text{Im } \gamma(t) > 0$ for $t > 0$, have at least 4-order tangency at the origin to the straight line under the angle $\frac{\pi}{2}(1 - \beta)$, $-1 < \beta < 1$, to the real axis \mathbb{R} , and let $f(z, t)$ map $\mathbb{H} \setminus \gamma_t$ onto \mathbb{H} and solve the chordal Löwner equation (1). Then*

$$\lim_{t \rightarrow +0} \frac{m_1(t)}{m_2(t)} = \frac{1 + \beta}{1 - \beta},$$

where

$$m_k(t) := \omega(f^{-1}(i, t); \gamma_{kt}, \mathbb{H} \setminus \gamma_t), \quad k = 1, 2,$$

γ_{1t} is the left side of γ_t , and γ_{2t} is the right side of γ_t .

The most important argument in the proof is the comparison of asymptotic parametric representations of γ_t in t and s at $s = 0$. This approach can be compared with the result by Earle and Epstein [4].

In Section 2, we solve a similar problem for a curve γ_t which has at least 6-order tangency with a circular arc in $\mathbb{H} \cup \{0\}$ tangential to \mathbb{R} at the origin. Since a scaling time change $t \rightarrow \alpha^2 t$ in the Löwner equation (1) is accompanied by changing $\lambda(t) \rightarrow \frac{1}{\alpha} \lambda(\alpha^2 t)$ and $f(z, t) \rightarrow \frac{1}{\alpha} f(\alpha z, \alpha^2 t)$, $\alpha \in \mathbb{R}$, we can assume without loss of generality that the circular arc is of radius 1, and the argument of its points is increasing when going from 0. We prove the following theorem.

Theorem 2. *Let $\gamma_t \in C^6$, $\gamma(0) = 0$, $\text{Im } \gamma(t) > 0$ and $\text{Re } \gamma(t) > 0$ for $t > 0$, have at least 6-order tangency at the origin to the circular arc of radius 1 centered at i , and let $f(z, t)$ map $\mathbb{H} \setminus \gamma_t$ onto \mathbb{H} and solve the chordal Löwner equation (1). Then*

$$\lim_{t \rightarrow +0} \frac{M_1^2(t)}{M_2(t)} = 2\pi,$$

where

$$M_k(t) := \omega(f^{-1}(i, t); \gamma_{kt}, \mathbb{H} \setminus \gamma_t), \quad k = 1, 2,$$

γ_{1t} is the left side of γ_t , and γ_{2t} is the right side of γ_t .

1. PROOF OF THEOREM 1

Proof of Theorem 1. For $\beta = 0$, Theorem 1 has been proved [5]. The cases $\beta > 0$ and $\beta < 0$ are symmetric to each other, and we will stop only on $\beta > 0$.

The Löwner equation (1) can be integrated in quadratures in particular cases [6]. For example, if $\lambda(t) = c\sqrt{t}$, $c \geq 0$, then a solution $f_c(z, t)$ to equation (1) maps $\mathbb{H} \setminus \gamma_t$ onto \mathbb{H} where γ_t is parameterized as

$$\gamma[0, t] = \{z = B\sqrt{x} : 0 \leq x \leq t\},$$



with $B = B(c) = |B(c)|e^{i\theta(c)}$,

$$|B(c)| = 2 \left(\frac{\sqrt{c^2 + 16} + c}{\sqrt{c^2 + 16} - c} \right)^{\frac{c}{2\sqrt{c^2 + 16}}}, \quad \theta(c) = \frac{\pi}{2} \left(1 - \frac{c}{\sqrt{c^2 + 16}} \right).$$

Suppose that a C^4 -slit γ_t satisfies the conditions of Theorem 1. Then there exists a driving function $\lambda(t) \in \text{Lip}(\frac{1}{2})$ such that a solution $w = f(z, t)$ to equation (1) maps $\mathbb{H} \setminus \gamma_t$ onto \mathbb{H} . For the arc-length parameter s , $\gamma(t(s))$ is represented as

$$\gamma(t(s)) = e^{i\theta} s + o(s^4), \quad s \rightarrow +0. \tag{2}$$

Denote

$$I_\theta(t) = \{xe^{i\theta} : 0 \leq x \leq t\}, \quad 0 < \theta < \frac{\pi}{2}, \quad t > 0.$$

There is $c > 0$ such that

$$\beta = \frac{c}{\sqrt{c^2 + 16}} \tag{3}$$

for which $\theta = \theta(c) = \frac{\pi}{2}(1 - \beta)$. Then $f_c(z, \tau)$ maps $\mathbb{H} \setminus I_{\theta(c)}(|B(c)|\sqrt{\tau})$ onto \mathbb{H} . The length $\sigma(\tau)$ of $I_{\theta(c)}(|B(c)|\sqrt{\tau})$ and the half-plane capacity τ of $I_{\theta(c)}(|B(c)|\sqrt{\tau})$ are related by

$$\sigma(\tau) = |B(c)|\sqrt{\tau}, \quad \tau > 0. \tag{4}$$

Let s denote the length of $\gamma_{t(s)}$ and let σ denote the length of projection of $\gamma_{t(s)}$ onto $I_{\theta(c)}(T)$. There is a C^4 -dependence $s = s(\sigma)$,

$$s(0) = 0, \quad s'(0) = 1, \quad s^{(k)}(0) = 0, \quad k = 2, 3, 4.$$

Therefore,

$$s = \sigma + o(\sigma^4), \quad \sigma \rightarrow +0. \tag{5}$$

Asymptotic expansion (2) implies an asymptotic behavior of a distance between γ_t and its projection on I_θ ,

$$\text{dist}(\gamma_{t(s)}, I_{\theta(c)}(\sigma(s))) = o(\sigma^4(s)), \quad s \rightarrow +0.$$

Lind, Marshall and Rohde [2] studied the closeness of half-plane capacities for two curves which are close together. According to Lemma 4.10 [2], we have that

$$t(s) - \tau(\sigma(s)) = o(s^2), \quad s \rightarrow +0,$$

where $\sigma(\tau)$ is given by (4). Hence, due to (4) and (5),

$$t(s(\sigma)) = \tau(\sigma) + o(s^2(\sigma)) = \tau(\sigma) + o(\sigma^2) = |B(c)|^{-2}s^2 + o(s^2), \quad s \rightarrow +0.$$

Take into account (2) and rewrite the last relation in the form

$$\gamma(t) = |B(c)|\sqrt{t} + \alpha(t)\sqrt{t}, \quad \lim_{t \rightarrow +0} \alpha(t) = 0. \tag{6}$$

Choose an arbitrary sequence $\{x_n\}$ of positive numbers $x_n, x_n \rightarrow \infty$ as $n \rightarrow \infty$, and denote

$$z = g_n(w, t) := \sqrt{x_n} f^{-1} \left(\frac{w}{\sqrt{x_n}}, \frac{t}{x_n} \right), \quad n = 1, 2, \dots \tag{7}$$

The function $z = f^{-1}(w, t)$ maps \mathbb{H} onto $\mathbb{H} \setminus \gamma[0, t]$, $f^{-1}(\lambda(t), t) = \gamma(t)$. So the functions $g_n(w, t)$ map \mathbb{H} onto $\mathbb{H} \setminus \gamma^{(n)}[0, t]$ where

$$\gamma^{(n)}(t) = \sqrt{x_n} \gamma \left(\frac{t}{x_n} \right) = e^{i\theta(c)} |B(c)| \sqrt{t} + \alpha \left(\frac{t}{x_n} \right) \sqrt{t},$$

$$g_n \left(\sqrt{x_n} \lambda \left(\frac{t}{x_n} \right), t \right) = \gamma^{(n)}(t), \quad 0 < t \leq T.$$



We see that

$$\gamma^{(n)}(t) - e^{i\theta(c)}|B(c)|\sqrt{t} = \alpha\left(\frac{t}{x_n}\right)\sqrt{t} \rightarrow 0, \quad n \rightarrow \infty,$$

and the convergence is uniform with respect to $t \in [0, T]$.

The Radó theorem [7], see also [8, p. 60], states that a sequence $\{h_n\}$ of conformal mappings h_n from the unit disk \mathbb{D} onto simply connected domains D_n bounded by Jordan curves ∂D_n , $0 \in D_n$, $h_n(0) = 0$, $h'_n(0) > 0$, converges uniformly on the closure of \mathbb{D} to $h : \mathbb{D} \rightarrow D$, ∂D is bounded by a Jordan curve, if and only if D_n converges to the kernel D and, for every $\epsilon > 0$, there exists $N > 0$ such that, for all $n > N$, there is a one-to-one correspondence $z_n : \partial D_n \rightarrow \partial D$, $|z_n(\zeta) - \zeta| < \epsilon$, $\zeta \in \partial D_n$. Markushevich [9] generalized the Radó theorem to domains with arbitrary boundaries.

Apply the Radó–Markushevich theorem to $g_n \circ p$ with a conformal mapping p from \mathbb{D} onto \mathbb{H} and obtain that the sequence $\{g_n(w, t)\}$ converges to $f_c^{-1}(w, t)$ as $n \rightarrow \infty$ uniformly on compact subsets of $\mathbb{H} \cup \mathbb{R}$.

Denote by $\Gamma_1[0, \tau(t)]$ the *left* side of the segment $I_{\theta(c)}(|B(c)|\sqrt{\tau})$ and denote by $\Gamma_2[0, \tau(t)]$ the *right* side of this segment. Similarly, denote $\gamma_{1n}[0, t]$ the *left* side of $\gamma^{(n)}[0, t]$, and denote $\gamma_{2n}[0, t]$ the *right* side of $\gamma^{(n)}[0, t]$. The functions $g_n^{-1}(z, t)$ map $\gamma_{1n}[0, t]$ and $\gamma_{2n}[0, t]$ onto segments $I_{1n} = I_{1n}(t) \subset \mathbb{R}$ and $I_{2n} = I_{2n}(t) \subset \mathbb{R}$, respectively. It is known, see, e.g. [5], that slit sides $\Gamma_1[0, \tau(t)]$ and $\Gamma_2[0, \tau(t)]$ are mapped by $f_c(z, t)$ onto

$$I_1 = I_1(t) = \left[\frac{c - \sqrt{c^2 + 16}}{2} \sqrt{t}, c\sqrt{t} \right] \quad \text{and} \quad I_2 = I_2(t) = \left[c\sqrt{t}, \frac{c + \sqrt{c^2 + 16}}{2} \sqrt{t} \right],$$

respectively. The uniform convergence of g_n to f_c^{-1} implies that $I_{1n}(t)$ tend to $I_1(t)$, and $I_{2n}(t)$ tend to $I_2(t)$ as $n \rightarrow \infty$.

Denote by $\gamma'_{1n}[0, t]$ and $\gamma'_{2n}[0, t]$ the *left* and the *right* sides of $\gamma[0, \frac{t}{x_n}]$, respectively. The function $f(z, \frac{t}{x_n})$ maps slit sides $\gamma'_{1n}[0, t]$ and $\gamma'_{2n}[0, t]$ onto segments $I'_{1n} = I'_{1n}(t) \subset \mathbb{R}$ and $I'_{2n} = I'_{2n}(t) \subset \mathbb{R}$, respectively. Compare $I_{kn}(t)$ and $I'_{kn}(t)$ by (7) and conclude that $I_{kn}(t) = \sqrt{x_n}I'_{kn}(t)$, and so

$$\text{meas } I_{kn}(t) = \sqrt{x_n} \text{meas } I'_{kn}(t), \quad k = 1, 2, \quad n \geq 1, \quad 0 < t \leq T.$$

The harmonic measure is invariant under conformal transformations. This gives that

$$\frac{m_1(\frac{t}{x_n})}{m_2(\frac{t}{x_n})} = \frac{\omega(f^{-1}(i, \frac{t}{x_n}), \gamma_1[0, \frac{t}{x_n}], \mathbb{H} \setminus \gamma(\frac{t}{x_n}))}{\omega(f^{-1}(i, \frac{t}{x_n}), \gamma_2[0, \frac{t}{x_n}], \mathbb{H} \setminus \gamma(\frac{t}{x_n}))} = \frac{\omega(i, I'_{1n}(t), \mathbb{H})}{\omega(i, I'_{2n}(t), \mathbb{H})}.$$

For $k = 1, 2$, $n \geq 1$, the harmonic measure $\omega(i; I'_{kn}(t), \mathbb{H})$ of $I'_{kn}(t)$ at i with respect to \mathbb{H} equals the angle divided over π under which the segment $I'_{kn}(t)$ is seen from the point i . Similarly, the harmonic measure $\omega(i; I_{kn}(t), \mathbb{H})$ of $I_{kn}(t)$ at i with respect to \mathbb{H} equals the angle divided over π under which the segment $I_{kn}(t)$ is seen from the point i , see, e.g. [8, p. 334]. This shows that the last term in the chain of equalities has a limit as $n \rightarrow \infty$, and

$$\lim_{n \rightarrow \infty} \frac{\omega(i, I'_{1n}(t), \mathbb{H})}{\omega(i, I'_{2n}(t), \mathbb{H})} = \lim_{n \rightarrow \infty} \frac{\tan(\pi\omega(i, I'_{1n}(t), \mathbb{H}))}{\tan(\pi\omega(i, I'_{2n}(t), \mathbb{H}))} = \lim_{n \rightarrow \infty} \frac{\text{meas } I'_{1n}(t)}{\text{meas } I'_{2n}(t)} = \lim_{n \rightarrow \infty} \frac{\text{meas } I_{1n}(t)}{\text{meas } I_{2n}(t)}.$$

This limit exists for every sequence $\{x_n\}$ tending to infinity. So there exists a limit for the ratio of $m_1(t)$ and $m_2(t)$ as $t \rightarrow +0$, and

$$\lim_{t \rightarrow +0} \frac{m_1(t)}{m_2(t)} = \lim_{n \rightarrow \infty} \frac{m_1(\frac{t}{x_n})}{m_2(\frac{t}{x_n})} = \lim_{t \rightarrow +0} \frac{\text{meas } I_1(t)}{\text{meas } I_2(t)} = \frac{\sqrt{c^2 + 16} + c}{\sqrt{c^2 + 16} - c} = \frac{1 + \beta}{1 - \beta},$$

where β is given by (3). This leads to the conclusion desired in Theorem 1 and completes the proof.

2. PROOF OF THEOREM 2

Proof of Theorem 2. The Löwner equation (1) admits an explicit integration [10] in the case when $\gamma_0[0, t]$ is a circular arc centered at i , $\gamma_0(0) = 0$, with an implicitly given driving function $\lambda(t)$. To be concrete, we will consider $\gamma_0[0, t]$ such that the argument of $(\gamma_0[0, t] - i)$ increases in t . Let a



solution $f_0(z, t)$ to equation (1) map $\mathbb{H} \setminus \gamma_0[0, t]$ onto \mathbb{H} . Its inverse $f_0^{-1}(w, t)$ is represented [10] by the Christoffel – Schwarz integral

$$\frac{1}{f_0^{-1}(w, t)} = \int_0^{\frac{1}{w}} \frac{(1 - \lambda_0 w)dw}{(1 - \beta_1 w)^2(1 - \beta_2 w)} = \frac{1}{2\pi} \log \frac{w - \beta_1}{w - \beta_2} + \frac{\beta_2 + \beta_1}{\beta_2 - \beta_1} \frac{1}{w - \beta_1},$$

where $\beta_1 = \beta_1(t)$ and $\beta_2 = \beta_2(t)$ are expanded in powers of $\sqrt[3]{t}$,

$$\beta_1(t) = A_1 \sqrt[3]{t^2} + A_2 t + A_3 \sqrt[3]{t^4} + \dots, \quad A_1 = -\frac{\sqrt[3]{9}}{\sqrt[3]{4\pi}},$$

and

$$\beta_2(t) = B_1 \sqrt[3]{t} + B_2 \sqrt[3]{t^2} + \dots, \quad B_1 = \sqrt[3]{12\pi}.$$

The driving function $\lambda_0(t)$ is evaluated by

$$\lambda_0(t) = 2\beta_1(t) + \beta_2(t) = C_1 \sqrt[3]{t} + C_2 \sqrt[3]{t^2} + \dots, \quad C_1 = B_1.$$

Suppose that a C^6 -slit γ_t satisfies the conditions of Theorem 2. Then there exists a driving function $\lambda(t)$ such that a solution $w = f(z, t)$ to equation (1) maps $\mathbb{H} \setminus \gamma_t$ onto \mathbb{H} . For the arc-length parameter s , represent a transformation of $\gamma(t(s))$,

$$\tilde{\gamma}(s) := \frac{2\gamma(t(s))}{2 + i\gamma(t(s))} = s + o(s^6), \quad s \rightarrow +0. \tag{8}$$

The function $f_0(z, \tau)$ maps $\mathbb{H} \setminus \gamma_0[0, \tau]$ onto \mathbb{H} . Hence,

$$G_0(w, \tau) = \frac{2f_0^{-1}(w, \tau)}{2 + if_0^{-1}(w, \tau)}$$

maps \mathbb{H} onto the exterior of the disk of radius 1 centered at $(-i)$ and slit along the segment $[0, \sigma] \subset \mathbb{R}$. The length $\sigma(\tau)$ of $[0, \sigma]$ and the half-plane capacity τ of $\gamma_0[0, \tau]$ are related by

$$\sigma(\tau) = \frac{2f_0^{-1}(\lambda_0(\tau), \tau)}{2 + if_0^{-1}(\lambda_0(\tau), \tau)} = B_1 \sqrt[3]{\tau} + o(\sqrt[3]{\tau^2}), \quad \tau \rightarrow +0. \tag{9}$$

Let s denote the length of $\tilde{\gamma}[0, s]$, and let σ denote the length of projection of $\tilde{\gamma}[0, s]$ onto $[0, \sigma]$. There is a C^6 -dependence $s = s(\sigma)$,

$$s(0) = 0, \quad s'(0) = 1, \quad s^{(k)}(0) = 0, \quad k = 2, \dots, 6.$$

Therefore,

$$s = \sigma + o(\sigma^6), \quad \sigma \rightarrow +0. \tag{10}$$

Asymptotic expansion (8) implies an asymptotic behavior of a distance between $\tilde{\gamma}$ and its projection on $[0, \sigma]$,

$$\text{dist}(\tilde{\gamma}[0, s], [0, \sigma(s)]) = o(\sigma^6(s)), \quad s \rightarrow +0.$$

According to Lemma 4.10 [2], we have that

$$t(s) - \tau(\sigma(s)) = o(s^3), \quad s \rightarrow +0,$$

where $\sigma(\tau)$ is given by (9). Hence, due to (9) and (10),

$$t(s(\sigma)) = \tau(\sigma) + o(s^3(\sigma)) = \tau(\sigma) + o(\sigma^3) = B_1^{-3} s^3 + o(s^3), \quad s \rightarrow +0.$$

Take into account (8) and rewrite the last relation in the form

$$\gamma(t) = B_1 \sqrt[3]{t} + \alpha(t) \sqrt[3]{t}, \quad \lim_{t \rightarrow +0} \alpha(t) = 0. \tag{11}$$



Choose an arbitrary sequence $\{x_n\}$ of positive numbers $x_n, x_n \rightarrow \infty$ as $n \rightarrow \infty$, and denote

$$z = g_n(w, t) := \sqrt{x_n} f^{-1} \left(\frac{w}{\sqrt{x_n}}, \frac{t}{x_n} \right), \quad n = 1, 2, \dots \quad (12)$$

The function $z = f^{-1}(w, t)$ maps \mathbb{H} onto $\mathbb{H} \setminus \gamma[0, t], f^{-1}(\lambda(t), t) = \gamma(t)$. So the functions

$$G_n(w, t) := \frac{2g_n(w, t)}{2 + ig_n(w, t)}, \quad n = 1, 2, \dots,$$

map \mathbb{H} onto the exterior of the disk of radius 1 centered at $(-i)$ minus $\tilde{\gamma}^{(n)}(t)$ where

$$\tilde{\gamma}^{(n)}(t) = \sqrt{x_n} \gamma \left(\frac{t}{x_n} \right) = B_1 \sqrt[3]{t} + \alpha \left(\frac{t}{x_n} \right) \sqrt[3]{t},$$

$$G_n \left(\sqrt{x_n} \lambda \left(\frac{t}{x_n} \right), t \right) = \tilde{\gamma}^{(n)}(t), \quad 0 < t \leq T.$$

We see that

$$\tilde{\gamma}^{(n)}(t) - B_1 \sqrt[3]{t} = \alpha \left(\frac{t}{x_n} \right) \sqrt[3]{t} \rightarrow 0, \quad n \rightarrow \infty,$$

and the convergence is uniform with respect to $t \in [0, T]$.

Apply the Radó – Markushevich theorem to $g_n \circ p$ with a conformal mapping p from \mathbb{D} onto \mathbb{H} and obtain that the sequence $\{G_n(w, t)\}$ converges to $G_0(w, t)$ which implies that $\{g_n(w, t)\}$ converges to $f_0^{-1}(w, t)$ as $n \rightarrow \infty$ uniformly on compact subsets of $\mathbb{H} \cup \mathbb{R}$.

Denote by $\Gamma_1[0, \tau(t)]$ the *left* side of the circular arc $\gamma_0[0, \tau]$ and denote by $\Gamma_2[0, \tau(t)]$ the *right* side of this circular arc. Similarly, denote $\gamma_{1n}[0, t]$ the *left* side of

$$\gamma^{(n)}[0, t] := \frac{2\tilde{\gamma}[0, t]}{2 - i\tilde{\gamma}[0, t]}$$

and denote $\gamma_{2n}[0, t]$ the *right* side of $\gamma^{(n)}[0, t]$. The functions $g_n^{-1}(z, t)$ map $\gamma_{1n}[0, t]$ and $\gamma_{2n}[0, t]$ onto segments $I_{1n} = I_{1n}(t) \subset \mathbb{R}$ and $I_{2n} = I_{2n}(t) \subset \mathbb{R}$, respectively. It is shown [10] that slit sides $\Gamma_1[0, \tau(t)]$ and $\Gamma_2[0, \tau(t)]$ are mapped by $f_0(z, t)$ onto

$$I_1 = I_1(t) = [\beta_1(t), \lambda_0(t)] \quad \text{and} \quad I_2 = I_2(t) = [\lambda_0(t), \beta_2(t)],$$

respectively. The uniform convergence of g_n to f_0^{-1} implies that $I_{1n}(t)$ tend to $I_1(t)$, and $I_{2n}(t)$ tend to $I_2(t)$ as $n \rightarrow \infty$.

Denote by $\gamma'_{1n}[0, t]$ and $\gamma'_{2n}[0, t]$ the *left* and the *right* sides of $\gamma[0, \frac{t}{x_n}]$, respectively. The function $f(z, \frac{t}{x_n})$ maps slit sides $\gamma'_{1n}[0, t]$ and $\gamma'_{2n}[0, t]$ onto segments $I'_{1n} = I'_{1n}(t) \subset \mathbb{R}$ and $I'_{2n} = I'_{2n}(t) \subset \mathbb{R}$, respectively. Compare $I_{kn}(t)$ and $I'_{kn}(t)$ by (12) and conclude that $I_{kn}(t) = \sqrt{x_n} I'_{kn}(t)$, and so

$$\text{meas } I_{kn}(t) = \sqrt{x_n} \text{meas } I'_{kn}(t), \quad k = 1, 2, \quad n \geq 1, \quad 0 < t \leq T.$$

The harmonic measure is invariant under conformal transformations. This gives that

$$\frac{M_1^2 \left(\frac{t}{x_n} \right)}{M_2 \left(\frac{t}{x_n} \right)} = \frac{\omega^2 \left(f^{-1} \left(i, \frac{t}{x_n} \right), \gamma_1 \left[0, \frac{t}{x_n} \right], \mathbb{H} \setminus \gamma \left(\frac{t}{x_n} \right) \right)}{\omega \left(f^{-1} \left(i, \frac{t}{x_n} \right), \gamma_2 \left[0, \frac{t}{x_n} \right], \mathbb{H} \setminus \gamma \left(\frac{t}{x_n} \right) \right)} = \frac{\omega^2 \left(i, I'_{1n}(t), \mathbb{H} \right)}{\omega \left(i, I'_{2n}(t), \mathbb{H} \right)}.$$

For $k = 1, 2, n \geq 1$, the harmonic measure $\omega(i; I'_{kn}(t), \mathbb{H})$ of $I'_{kn}(t)$ at i with respect to \mathbb{H} equals the angle divided over π under which the segment $I'_{kn}(t)$ is seen from the point i . Similarly, the harmonic measure $\omega(i; I_{kn}(t), \mathbb{H})$ of $I_{kn}(t)$ at i with respect to \mathbb{H} equals the angle divided over π under which the segment $I_{kn}(t)$ is seen from the point i . This shows that the last term in the chain of equalities has a limit as $n \rightarrow \infty$ and

$$\lim_{n \rightarrow \infty} \frac{\omega^2 \left(i, I'_{1n}(t), \mathbb{H} \right)}{\omega \left(i, I'_{2n}(t), \mathbb{H} \right)} = \lim_{n \rightarrow \infty} \frac{\tan^2 \left(\pi \omega \left(i, I'_{1n}(t), \mathbb{H} \right) \right)}{\tan \left(\pi \omega \left(i, I'_{2n}(t), \mathbb{H} \right) \right)} = \lim_{n \rightarrow \infty} \frac{\text{meas}^2 I'_{1n}(t)}{\text{meas } I'_{2n}(t)} = \lim_{n \rightarrow \infty} \frac{\text{meas}^2 I_{1n}(t)}{\text{meas } I_{2n}(t)}.$$



This limit exists for every sequence $\{x_n\}$ tending to infinity. So there exists a limit for the ratio of $M_1^2(t)$ and $M_2(t)$ as $t \rightarrow +0$, and

$$\begin{aligned} \lim_{t \rightarrow +0} \frac{M_1^2(t)}{M_2(t)} &= \lim_{n \rightarrow \infty} \frac{M_1^2(\frac{t}{n})}{M_2(\frac{t}{n})} = \lim_{t \rightarrow +0} \frac{\text{meas}^2 I_1(t)}{\text{meas} I_2(t)} = \lim_{t \rightarrow +0} \frac{(\lambda_0(t) - \beta_1(t))^2}{\beta_2(t) - \lambda_0(t)} = \\ &= \lim_{t \rightarrow +0} \frac{(B_1 \sqrt[3]{t} + (C_2 - A_1) \sqrt[3]{t^2} + \dots)^2}{(B_2 - C_2) \sqrt[3]{t^2} + \dots} = \frac{B_1^2}{B_2 - C_2} = \frac{B_1^2}{-2A_1} = 2\pi. \end{aligned}$$

This leads to the conclusion desired in Theorem 2 and completes the proof.

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Асимптотическое отношение гармонических мер сторон разреза

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Статья посвящена геометрии решений хордового уравнения Левнера, основанной на сравнении сингулярных решений и гармонических мер берегов разреза в верхней полуплоскости, порожденного управляющей функцией. Найдено асимптотическое отношение гармонических мер берегов разреза, касательного к прямой под заданным углом, и разреза, имеющего высокий порядок касания к дуге окружности, касающейся действительной оси.

Ключевые слова: уравнение Левнера, сингулярное решение, гармоническая мера, емкость в полуплоскости.

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О НЕКОТОРЫХ ИНТЕГРАЛЬНЫХ СВОЙСТВАХ МОДИФИЦИРОВАННЫХ ФУНКЦИЙ БЕССЕЛЯ

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Представлены новые интегральные тождества для модифицированных функций Бесселя произвольного комплексного порядка. Изучены свойства интегральных преобразований Лебедева – Скальской.

Ключевые слова: модифицированные функции Бесселя комплексного порядка, интегральные преобразования Конторвича – Лебедева, интегральные преобразования Лебедева – Скальской.

1. НЕКОТОРЫЕ СВОЙСТВА ФУНКЦИЙ $\operatorname{Re} K_{\alpha+i\beta}(x)$ И $\operatorname{Im} K_{\alpha+i\beta}(x)$

Можем записать вещественную и мнимую часть модифицированных функций Бесселя комплексного порядка в виде

$$\operatorname{Re} K_{\alpha+i\beta}(x) = \frac{K_{\alpha+i\beta}(x) + K_{\alpha-i\beta}(x)}{2} \quad \text{и} \quad \operatorname{Im} K_{\alpha+i\beta}(x) = \frac{K_{\alpha+i\beta}(x) - K_{\alpha-i\beta}(x)}{2i},$$

где $K_\nu(x)$ — модифицированная функция Бесселя второго рода (также называемая функцией Макдональда).

Функции $K_{i\beta}(x)$, $\operatorname{Re} K_{\alpha+i\beta}(x)$ и $\operatorname{Im} K_{\alpha+i\beta}(x)$ имеют интегральные представления [1, 2]

$$K_{i\beta}(x) = \int_0^\infty e^{-x \cosh t} \cos(\beta t) dt,$$

$$\operatorname{Re} K_{\alpha+i\beta}(x) = \int_0^\infty e^{-x \cosh t} \cosh(\alpha t) \cos(\beta t) dt, \quad (1)$$

$$\operatorname{Im} K_{\alpha+i\beta}(x) = \int_0^\infty e^{-x \cosh t} \sinh(\alpha t) \sin(\beta t) dt. \quad (2)$$

Из (1), (2) следует, что возможно переписать $\operatorname{Re} K_{\alpha+i\beta}(x)$ в виде косинус-преобразования Фурье

$$\operatorname{Re} K_{\alpha+i\beta}(x) = \left(\frac{\pi}{2}\right)^{1/2} F_C[e^{-x \cosh t} \cosh(\alpha t); t \rightarrow \beta] \quad (3)$$