Известия Саратовского университета. Новая серия. Серия: Математика. Механика. Информатика. 2026. Т. 26, вып. 2. С. 175–186
Izvestiya of Saratov University. Mathematics. Mechanics. Informatics, 2026, vol. 26, iss. 2, pp. 175–186
https://mmi.sgu.ru DOI: https://doi.org/10.18500/1816-9791-2026-26-2-175-186
EDN: https://elibrary.ru/EVOAWM

Article

Merging of Euler's method with trigonometric functions for accurate ray path in a two-gradient medium

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Abstract. In this paper, an analytical solution is presented to provide an accurate trajectory of a ray propagating from the known position of the source to the receiver in a two-gradient medium. A system of two linear gradients connects two different layered media when the transition from one to the other occurs at some boundary. Within each medium, refractive indices determine the propagation of waves and, accordingly, the curved trajectories of rays. Different radii of curves make it difficult to track the ray as it propagates from the source to the receiver. Euler's method provides an exact solution for a one-gradient model. However, in the case of two gradients, the accurate solution cannot be obtained because of the underdetermined common system for ray curves and computational complexity. In this paper, a technique is described that combines Euler's method and trigonometric functions to derive direct formulas for calculating key angles responsible for the ray path in both gradient media. An exact solution overcomes the the drawbacks of iterative approaches, which are subject to computational errors. The basic formula developed for two-gradient models was tested using a small set of real data by transforming it into a particular case of a one-gradient model. The independence of the evaluations is confirmed by comparing the calculated parameters with those taken from an earlier publication. The derived formulas are essential for solving problems in oil and gas exploration, geothermal exploration, and other challenges related to energetics. The solution can be extended for acoustic, optical, and other tasks.

Keywords: ray tracing algorithm, two-gradient model, incidence angle, analytic solution

Acknowledgements: This work was supported by the Ministry of Science and Higher Education of the Russian Federation (project No. 125021302095-2 "Fundamental basis of innovative, digital technologies for forecasting, prospecting, exploration and development of oil and gas resources (fundamental, prospecting, applied, economic and interdisciplinary research until 2030)"). We are grateful to Kristin Vogfjord (Icelandic Meteorological Office, Reykjavik) for useful discussion about gradient models. Also, we thank Martin Hensch for providing us with the data.

For citation: Smaglichenko T. A. Merging of Euler's method with trigonometric functions for accurate ray path in a two-gradient medium. *Izvestiya of Saratov University. Mathematics. Mechanics. Informatics*, 2026, vol. 26, iss. 2, pp. 175–186. DOI: https://doi.org/10.18500/1816-9791-2026-26-2-175-186, EDN: EVOAWM

Научная статья
УДК 517.98

Объединение метода Эйлера и тригонометрических функций для определения точного пути луча в двухградиентной среде

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Аннотация. В данной работе представлено аналитическое решение, позволяющее получить точную траекторию луча, распространяющегося от известного положения источника к приёмнику в двухградиентной среде. Система из двух линейных градиентов соединяет две различные слоистые среды, переход от одной в другую происходит на некоторой границе. В каждой среде показатели преломления определяют распространение волн и, соответственно, искривлённые пути лучей. Различные радиусы кривых затрудняют отслеживание луча при его распространении от источника до приёмника. Метод Эйлера даёт точное решение для модели, характеризующейся одним градиентом. Однако в случае двух градиентов точное решение невозможно построить из-за недоопределённости общей системы кривых лучей и вычислительной сложности. В статье описывается метод, объединяющий метод Эйлера и тригонометрические функции для вывода прямых формул при расчете ключевых углов, отвечающих за траекторию луча в обеих градиентных средах. Точное решение позволяет преодолеть недостаток итерационного подхода, связанный с вычислительной погрешностью. Базовая формула, разработанная для двухградиентных моделей, была протестирована на небольшом наборе реальных данных путём её преобразования в частный случай одноградиентной модели. Независимость оценок подтверждена сравнением расчётных параметров с параметрами, взятыми из более ранней публикации. Полученные формулы имеют важное значение для решения проблем при разведке месторождений нефти и газа, геотермальных ресурсов и других, связанных с энергетикой. Решение может быть обобщено на акустические, оптические и другие задачи.

Ключевые слова: алгоритм трассировки лучей, двухградиентная модель, угол падения, аналитическое решение

Благодарности: Исследование выполнено при финансовой поддержке Министерства науки и высшего образования Российской Федерации (проект № 125021302095-2 «Фундаментальный базис инновационных, цифровых технологий прогноза, поиска, разведки и освоения нефтегазовых ресурсов (фундаментальные, поисковые, прикладные, экономические и междисциплинарные исследования до 2030 года)»). Авторы благодарят Кристин Фогфьорд (Исландское метеорологическое бюро, Рейкьявик) за полезное обсуждение градиентных моделей и Мартина Хенша за предоставленные данные.

Для цитирования: *Smaglichenko T. A.* Merging of Euler's method with trigonometric functions for accurate ray path in a two-gradient medium [*Смагличенко Т. А.* Объединение метода Эйлера и тригонометрических функций для определения точного пути луча в двухградиентной среде] // Известия Саратовского университета. Новая серия. Серия: Математика. Механика. Информатика. 2026. Т. 26, вып. 2. С. 175–186. DOI: https://doi.org/10.18500/1816-9791-2026-26-2-175-186, EDN: EVOAWM

Introduction

Gradient models are widely used in energy production areas. One example is the formalization of the process of water injection into wells at large oil fields. The process of water injection helps to extract oil from the depths to the surface. The amount of water injected is related to the amount of oil produced through a linear relationship that includes a gradient, the numerical value of which characterizes the reservoir pressure [1]. The gradient model considered in this paper is also based on linear dependence. It is determined by seismic velocity growth with depth and changing gradients. Thus, constructing optimal gradient models is part of production processes.

It is essential to recognize that the complexity of geological structures extends beyond one gradient. Oil companies report difficulties in extracting oil and gas from deep-seated traps [2]. Moreover, the lack of knowledge of geological structures can lead to accidents and equipment failure [3]. Such a case occurred during the drilling of geothermal wells in Iceland. At a certain boundary, the velocity gradient was altered under the influence of hot magma. The equipment failed. This event slowed down the energy extraction process for several years.

The best minimal model for a one-gradient medium can be found by applying the method developed by the authors of the article [4]. The technique is effective for detecting gas areas. Local earthquake data from neighboring areas of the Icelandic region (Subarctic zone) were used. The result revealed that the P-wave velocity gradient for an area where up to 550 tons of carbon dioxide are extracted has a greater numerical value than the values of other areas. The gradient increase began at a depth of 2.5 km and increased to 6.0 km [5].The authors (participants of the project RANNIS ID-152432-051) discussed the issue of the linear gradient distribution at a meeting with representatives of the Icelandic Meteorological Office (IMO). According to their information, in many areas of Iceland, the second velocity gradient represents a change in velocity at a depth of 3.0 km. This has led to the need to consider the two-gradient model.

Another motivation for such a model is the existence of the boundary of velocity change in the Arctic regions on Franz Josef Land [6]. The depth at which properties of geological material changed was determined based on knowledge of the output angle of a seismic ray at the surface. This angle was calculated using readings from seismographs and galvanometers tuned to different wavelengths. The instrumental data were stored at the expeditionary station named "Arkticheskaya" (the Research Institute of Arctic Geology) and the station "Kheys" (Krenkel Polar Station, Kheysa Island) located on Alexandra Land Island. The coordinates of the epicenters were taken from the bulletins of the USSR seismic station network. Using these data, the author of the work [7] constructed experimental curves and concluded that the wave changes its direction at a depth of 2.5–3.0 km. Thus, it was assumed that the jump in velocity occurs in this depth range. To determine the formula that establishes the relation between the output angle and "apparent" velocity at different depths, the author of [8] developed a differentiated method based on two formulas of Benndorf and Wiechert – Chibisov.

Thus, the output angle, velocity, and depth are analytically connected parameters. From the history of seismology [6–8], we can note that instrumental observations of the output angles permitted the construction of velocity models. Nowadays, knowledge about the velocity distribution with depth at different areas is stable. Another task becomes relevant. Instead of expensive equipment, mathematical modeling should be used to calculate the incidence angle of a seismic ray at the surface in a complex velocity gradient medium. Note that in exploration methods, the incidence angle is the central parameter determining the residual between theoretical and experimental values. Creating models of geological structures helps to develop digital twins that provide energy savings in the manufacturing process. According to oil and gas companies operating in the Arctic, the main reserves are confined to the boundary depths (about 3.0 km) of the Achimov and Jurassic deposits [2]. Other examples are described above. Thus, the properties of rocks change sharply at a certain boundary. The different structures of geological material may correspond to two gradients of seismic velocity.

A medium with a velocity discontinuity between two linear gradients is referred to as a two-gradient model. Different parameters characterize each linear gradient. Curved ray trajectories represent a system of two circles with centers at various points of the medium. Knowledge of the incidence angle at the receiver determine the positions of ray paths under the condition of an accurate fit of the ray to the receiver when the coordinates of the source and receiver are known. Therefore, the goal is to develop an analytical solution for determining a formula to calculate the incidence angle on the Earth's surface and other key angles in a medium with two linear gradients. Note that the boundary, at which a velocity discontinuity occurs, is assumed to be given.

Other approaches to constructing ray paths are based on the "shooting" method. The ray is emitted from the source, refracted at specified boundary surfaces, and reaches the station. One such procedure is the bending (pseudo-bending) method, in which curved rays are emitted from the source to hit the receiver. However, the solution is unstable [9].

Therefore, the author of the work [10] developed a parameterized method of "shooting" for the Cartesian coordinate system, which assumes that direct rays come from the source. The increase in the accuracy of the entrance at the receiver was achieved because the approximate calculations of the method [9] were replaced in [10] by integral analytical solutions. In [11] and [12], the convergence control was developed. Owing to this approach, the degree of convergence was improved by 10%. The authors of [13] revised the method by using spherical coordinates.

The focus of this paper is on accurately determining the ray path, specifically the method that determines the precise path of the ray from the given source to the receiver. For a one-gradient medium, the solution of the system of differential Euler equations provides the calculation of coordinates for the ray's point if the incident angle is known [14]. The solution cannot be used when rays pass through the given boundary of the gradient change.

The authors of [15] constructed the solution for a simple model consisting of two basic layers. The first is located from the surface to the upper bound of the known depth. A constant velocity of wave propagation characterizes this layer. The linear gradient of velocity describes the second layer. A variational algorithm that solves the Cauchy problem and applies the iterative Runge – Kutta technique was used to determine the solution. However, there is no ray-tracing solution for two-gradient media.

The issue is that the solution for the common ray path is not obtained by using Euler's formula for both the medium in the upper layer and the medium below the boundary. The tangents to the ray curve are distributed like a fan, and there are multiple refractions in the limits of each gradient layer. To find the connection between two different parts of the model, one can use only the tangent to the ray that falls on the boundary at the point where the parameters of the first-gradient are changed when the ray accurately passes from the source to the receiver.

In the given paper, analytical expressions for the tracing angles have been obtained based on the proposed approach. Euler's method is combined with the algorithm for deriving trigonometric functions that assign the angles' values for exact ray trajectories in a two-gradient medium.

1. Statement of the problem

Suppose it is known from a priori information about the area being studied that the velocity undergoes a discontinuity at a certain depth. At the same time, there are assumptions about the velocity parameters, which describe the media above and below the boundary depth z_{bd}. Let a linear relationship characterize the medium of the first-gradient of velocity as $V(z) = V_0 + V_1 z$, while the medium of the second gradient can be described as $V(z) = \tilde{V}_0 + \tilde{V}_1 z$, where z stands for depth, V_0 and V_1 are the velocity value at the surface and the gradient value above the boundary depth, $\tilde{V}_0$ and $\tilde{V}_1$ are parameters for the second part of the medium. Figure 1, *a* demonstrates an image of two velocity gradients. The first-gradient is presented by various gray bars, while dark gray bars correspond to the second gradient. A bold line denotes the boundary between two gradients. Figure 1, *b* shows the velocity profile with a linear dependence of velocity on depth. For any depth, represented by a horizontal line in Fig. 1, *a*, the velocity is calculated using formulas in Fig. 1, *b*.

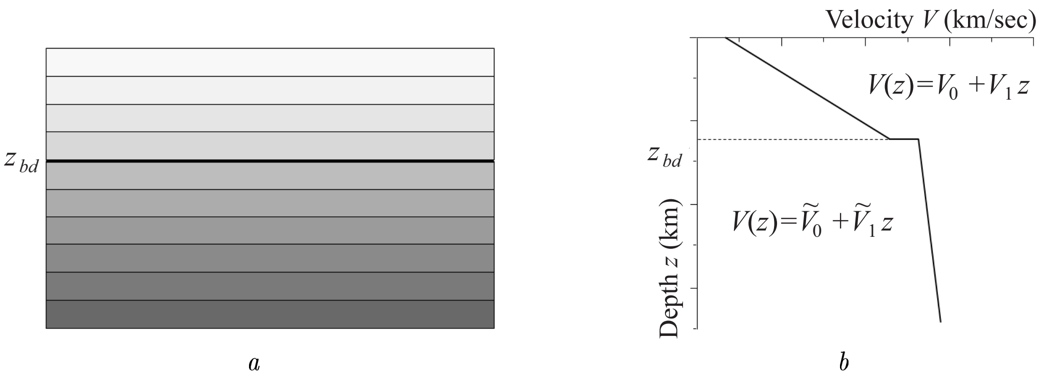

Fig. 1. Two-gradient model: *a* shows images of the gradients; *b* shows linear velocity functions

Figure 2 shows a ray path in the ray plane (w, z), where $w = \frac{x-\overline{x}}{\cos\varphi}$, $\overline{x}$ is the Cartesian coordinate of the receiver, and φ is the azimuth angle. A ray travels from the source (denoted by the closed circle) in the medium above the boundary depth. Then its trace continues in the medium of the second gradient. Finally, it reaches the receiver (denoted by the triangle) in the medium of the first-gradient.

Figure 3 illustrates an error if Snell's law is used for the ray tangents of the first-gradient medium refracted at the boundary between two media. In that case, the solution can be constructed; however, the path from the source to the receiver will not be exact.

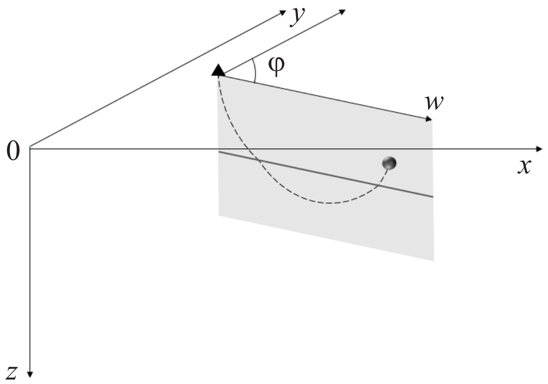

Fig. 2. The ray plane (w, z) within a Cartesian plane. Angle φ is the angle between the plane and the direction $0Y$, which is normally the north direction. A bold line denotes the gradient change

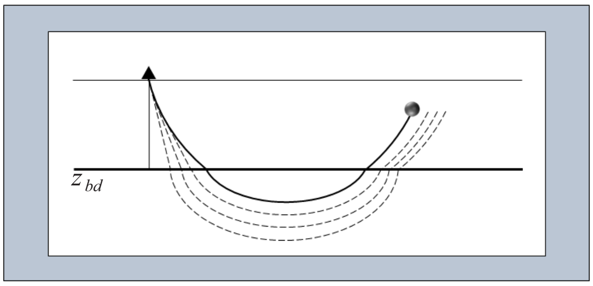

Fig. 3. The solution error occurs if Euler's method is combined with Snell's law for the ray tangents of the first-gradient medium

If the ray tangent refracts at the point, for which the ray path is accurate, then the number of unknown angle parameters will be greater than the number of equations. Therefore, in this paper, the algorithm has been developed to overcome this uncertainty.

Figure 4 shows the key angles that determine the ray trace from the source to the receiver. Two coordinate systems were used. The first is for the ray points in the medium above the boundary depth. The origin of this system coincides with the receiver point. The second system is in the medium below the boundary depth. The origin corresponds to the transition point of the gradient change for the exact ray path.

The problem is formulated as follows: Derive the formula to calculate α an incidence angle on the receiver through the parameters V_0 and V_1, $\tilde{V}_0$ and $\tilde{V}_1$, and the boundary depth z_{bd}.

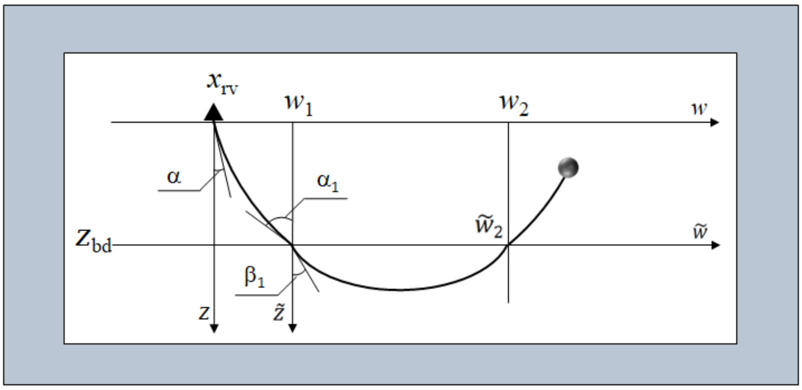

Fig. 4. A plane of the ray: α is the incidence angle of ray on the receiver, α_1 is the angle of incidence on the depth z_{bd} that is a boundary between two gradients, β_1 is the refraction angle, which determines the wave propagation in the medium of the second gradient, (w_1, z_{bd}) and (w_2, z_{bd}) are points of intersection of the ray with the boundary depth, when the coordinate system is used for the medium of the first-gradient; $(\tilde{w}_2, 0.0)$ is a point in the coordinate system of the medium of the second gradient

2. Derivation of the key angles formulas for a two-gradient model

According to [14], the application of Euler's method to the system of differentiated equations determines the trajectory of a seismic ray in a one-gradient medium:

$$y - \overline{y} = (x - \overline{x}) \tan \varphi, \tag{1}$$

$$(V_0 + V_1 z)^2 + \frac{{V_1}^2}{\cos^2\varphi} \left[x - \overline{x} - \frac{(V_0 + V_1 \overline{z}) \cos \varphi}{V_1 \tan \alpha} \right]^2 = \frac{(V_0 + V_1 \overline{z})^2}{\sin^2\alpha}. \tag{2}$$

The ray travels from the point $(\overline{x}, \overline{y}, \overline{z})$ to (x, y, z). In the following sections, we explain how equations (1) and (2) are applied and define the method to obtain the formula for the incidence angle on the receiver in a two-gradient medium.

2.1. Relationship between the incidence angle on the receiver and the incidence angle to the boundary depth

The formula to derive the angle α in the medium of the first-gradient can be obtained using equation (2) for a portion of the ray passing from point $(\overline{x}, \overline{z}) = (x_{rv}, z_{rv})$ to $(x, z) = (x_1, z_{bd})$, where (x_{rv}, z_{rv}) is the receiver coordinates, and x_1 is coordinate corresponding w_1 on Fig. 2:

$$(V_0 + V_1 z_{bd})^2 + \frac{{V_1}^2}{\cos^2\varphi} \left[x_1 - x_{rv} - \frac{(V_0 + V_1 z_{rv}) \cos \varphi}{V_1 \tan \alpha} \right]^2 = \frac{(V_0 + V_1 z_{rv})^2}{\sin^2\alpha}.$$

Let us perform operations inside the square brackets first. Since $z_{rv} = 0$, we have

$$(V_0 + V_1 z_{bd})^2 + \frac{{V_1}^2}{\cos^2\varphi} \left[(x_1 - x_{rv})^2 - 2\frac{(x_1 - x_{rv}) V_0 \cos \varphi}{V_1 \tan \alpha} + \frac{V_0^2 \cos^2\varphi}{V_1^2 \tan^2\alpha} \right] = \frac{V_0^2}{\sin^2\alpha}.$$

After multiplying, simplifying, and using trigonometric identities, we get

$$(V_0 + V_1 z_{bd})^2 + V_1^2 w_1^2 - 2 V_0 V_1 w_1 \frac{1}{\tan \alpha} + \frac{V_0^2}{\tan^2\alpha} = \frac{V_0^2}{\sin^2\alpha},$$

$$(V_0 + V_1 z_{bd})^2 + V_1^2 w_1^2 - \frac{2 V_0 V_1 w_1}{\tan \alpha} = \frac{V_0^2}{\sin^2\alpha} - \frac{V_0^2}{\tan^2\alpha} = V_0^2. \tag{3}$$

From equation (3), we obtain

$$\tan\alpha = \frac{2V_0V_1w_1}{(V_0+V_1z_{bd})^2+V_1^2w_1^2-V_0^2}. \tag{4}$$

The angle α_1 can be determined by applying equation (2) to the inverse trajectory of a ray from point $(\overline{x},\overline{z})=(x_1,z_{bd})$ to $(x,z)=(x_{rv},z_{rv})$:

$$\begin{gathered}(V_0+V_1z_{rv})^2+\frac{V_1{}^2}{\cos^2\varphi}\left[x_{rv}-x_1-\frac{(V_0+V_1z_{bd})\cos\varphi}{V_1\tan\alpha_1}\right]^2=\frac{(V_0+V_1z_{bd})^2}{\sin^2\alpha_1},\\ V_0^2+\frac{V_1{}^2}{\cos^2\varphi}\left[(x_{rv}-x_1)^2-2\frac{(x_{rv}-x_1)(V_0+V_1z_{bd})\cos\varphi}{V_1\tan\alpha_1}+\frac{(V_0+V_1z_{bd})^2\cos^2\varphi}{V_1^2\tan^2\alpha_1}\right]=\\ =\frac{(V_0+V_1z_{bd})^2}{\sin^2\alpha_1}.\end{gathered} \tag{5}$$

The equation (5) is transformed using steps similar to obtaining equation (4). Finally, we have the following formula:

$$\tan\alpha_1=\frac{2(V_0+V_1z_{bd})V_1w_1}{(V_0+V_1z_{bd})^2-V_1^2w_1^2-V_0^2}. \tag{6}$$

Next, we divide equation (4) by equation (6). After reductions and transformations, we get

$$\frac{\tan\alpha}{\tan\alpha_1}=\frac{V_0}{(V_0+V_1z_{bd})}\left[1-\frac{2V_1^2w_1^2}{(V_0+V_1z_{bd})^2+V_1^2w_1^2-V_0^2}\right]. \tag{7}$$

From formula (7), we have a formula for determining $\cot^2\alpha_1$:

$$\cot^2\alpha_1=\frac{V_0^2}{(V_0+V_1z_{bd})^2}\left[1-\frac{2V_1^2w_1^2}{(V_0+V_1z_{bd})^2+V_1^2w_1^2-V_0^2}\right]^2\frac{1}{\tan^2\alpha}. \tag{8}$$

2.2. Relationship between the angle of incidence on the boundary depth and the refraction angle

According to Snell's law for two media, the following equation links the incidence angle α_1 and refraction angle β_1 via the velocity parameters:

$$\frac{\sin\alpha_1}{\sin\beta_1}=\frac{(V_0+V_1z_{bd})}{(\tilde{V}_0+\tilde{V}_1z_{bd})}. \tag{9}$$

From equation (9), we obtain

$$\sin^2\beta_1=\sin^2\alpha_1\frac{(\tilde{V}_0+\tilde{V}_1z_{bd})^2}{(V_0+V_1z_{bd})^2}. \tag{10}$$

2.3. Obtaining the formula for the refraction angle in the medium of the second gradient

Let us set down a coordinate system with the origin at the point where the first-gradient is changed by the second for the ray path. Let us consider the medium of the second gradient and determine the formula for $\tan\beta_1$ in the plane $(\tilde{w},\tilde{z})$.

Using equation (2) for the portion of a ray going from point $(\overline{x},\overline{z})=(x_1,z_{bd})$ that has coordinates $(0.0, 0.0)$ in a new system into the point $(x,z)=(x_2,z_{bd})$ that has coordinates $(\tilde{x}, 0.0)$ we obtain

$$\tilde{V}_0^2+\frac{\tilde{V}_1{}^2}{\cos^2\varphi}\left[\tilde{x}_2-\frac{\tilde{V}_0\cos\varphi}{\tilde{V}_1\tan\beta_1}\right]^2=\frac{\tilde{V}_0^2}{\sin^2\beta_1}. \tag{11}$$

Note that

$$\frac{\tilde{x}_2^2}{\cos^2\varphi} = \frac{(\tilde{x}_2 - \overline{x})^2}{\cos^2\varphi} = \tilde{w}_2^2.$$

Then equation (11) can be written as

$$\tilde{V}_0^2 + \tilde{V}_1^2\tilde{w}_2^2 - 2\tilde{w}_2\tilde{V}_1\tilde{V}_0\frac{1}{\tan\beta_1} + \frac{\tilde{V}_0^2}{\tan^2\beta_1} = \frac{\tilde{V}_0^2}{\sin^2\beta_1}. \tag{12}$$

After transforming equation (12), we obtain the formula for $\tan\beta_1$:

$$\tan\beta_1 = \frac{2\tilde{V}_0}{\tilde{V}_1\tilde{w}_2}. \tag{13}$$

Let us apply the trigonometric equation

$$1 + \cot^2\beta_1 = \frac{1}{\sin^2\beta_1}. \tag{14}$$

Then from equations (13) and (14) we get

$$\sin^2\beta_1 = \frac{4\tilde{V}_0^2}{4\tilde{V}_0^2 + \tilde{V}_1^2\tilde{w}_2^2}. \tag{15}$$

2.4. The formula for the angle of incidence on the boundary depth via the parameters of two gradients

Let us equate the right-hand sides of the expressions obtained for $\sin^2\beta_1$, which are represented by equations (10) and (15). This technique will allow us to determine the angle α_1 through the parameters describing two gradients:

$$\sin^2\alpha_1 = \frac{4(V_0 + V_1 z_{bd})^2\tilde{V}_0^2}{(\tilde{V}_0 + \tilde{V}_1 z_{bd})^2(4\tilde{V}_0^2 + \tilde{V}_0^2\tilde{w}_2^2)}. \tag{16}$$

2.5. The formula to calculate the angle of incidence on the receiver

Let us apply the trigonometric equation (14) for the angle α_1. We will use equations (8) and (16). Substituting the obtained expressions for $\cot^2\alpha_1$ and $\sin^2\alpha_1$ we obtain

$$\begin{aligned} 1 + \frac{V_0^2}{(V_0 + V_1 z_{bd})^2}\left[1 - \frac{2V_1^2 w_1^2}{(V_0 + V_1 z_{bd})^2 - V_0^2 + V_1^2 w_1^2}\right]^2 \frac{1}{\tan^2\alpha} = \\ = \left(1 + \frac{\tilde{V}_1^2\tilde{w}_2^2}{4\tilde{V}_0^2}\right)\frac{(\tilde{V}_0 + \tilde{V}_1 z_{bd})^2}{(V_0 + V_1 z_{bd})^2}. \end{aligned} \tag{17}$$

After some transformation of equation (17), we determine the incidence angle at the receiver α as its trigonometric function $\tan\alpha$:

$$\tan\alpha = \frac{S_1}{S_2}, \tag{18}$$

where

$$S_1 = \frac{V_0}{(V_0 + V_1 z_{bd})}\left[1 - \frac{2V_1^2 w_1^2}{(V_0 + V_1 z_{bd})^2 - V_0^2 + V_1^2 w_1^2}\right]$$

and

$$S_2 = \sqrt{\left[\left(1 + \frac{\tilde{V}_1^2\tilde{w}_2^2}{4\tilde{V}_0^2}\right)\frac{(\tilde{V}_0 + \tilde{V}_1 z_{bd})^2}{(V_0 + V_1 z_{bd})^2} - 1\right]}. \tag{19}$$

3. Testing of the basic equation by application to a small set of real data

The reliability of the basic constructed equation (18) is confirmed by performing the following expertise. Let us use the result of ray tracing for a one-gradient model that was published a few years ago in [16]. The dataset contains local seismic events recorded in 2001 by the South Iceland Lowland (SIL) seismic network. The Icelandic Meteorological Office provided the data within the framework of a joint research project supported by the RANNIS fund ID-152432-051. Table 1 presents the parameters of the five rays, which involve the permanent stations (receivers) names, coordinates of the point of the ray maximal descent $(w_{\max}, z_{\max})$, the incidence angle to the receiver θ, the local earthquake depth, the epicentral distance is the distance from the epicenter (projection of the earthquake source on the Earth's surface) to the station [16].

Table 1

Ray path parameters calculated for the seismic data sample (after [16])

Station name	$z_{\max}$, km	$w_{\max}$, km	θ (radians)	Depth, km	Epicentral distance, km
Hau	12.4865770	17.2813225	0.319398791	4.4	32.4136772
Hei	17.1106834	22.0989037	0.253079265	3.7	42.8926048
Hau	11.3059006	16.0329971	0.342429072	1.9	31.2556458
Kro	14.6913977	19.5900269	0.283867538	1.5	38.6784210
Kro	15.3930874	20.3199387	0.274182767	2.9	39.5899811

Figure 5 shows the ray traces in the one plane. The parameters were calculated for a one-gradient model that is the SIL model, for which $V_0 = 4.926$, $V_1 = 0.479$ [4, 16].

Let us check the formula (18) by transforming it to the formula for a one-gradient model. To do this, we substitute the parameters of the first-gradient V_0 and V_1, instead of the parameters of the second gradient $\tilde{V}_0$ and $\tilde{V}_1$. Then, after the transformation of equation (19), we get the following equation to calculate the incidence angle to the receiver α as its trigonometric function $\tan\alpha$:

$$\tan\alpha = \frac{t_1}{t_2}, \tag{20}$$

where $t_1 = S_1$, $t_2 = \dfrac{V_1 \tilde{w}_2}{2V_0}$.

Note that in Table 1, the incidence angle to the receiver, denoted by θ after research [16], while in the given paper the same angle is denoted as α.

Let us assume that the boundary depth is equal to the source depth: $z_{bd} = z_{source}$ (Fig. 6). Then $\tilde{w}_2 = w_2 - w_1$, where w_2 is the epicentral distance, and $w_1 = 2w_{\max} - w_2$. Using the data from Table 1 and substituting them into equation (20), we calculate the incidence angle for each testing ray.

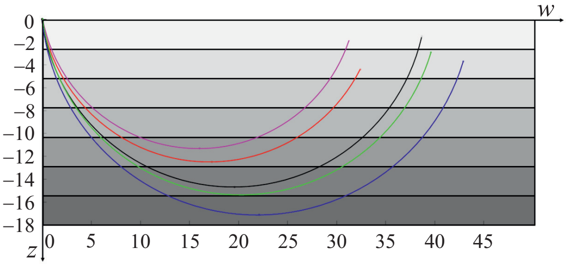

Fig. 5. Ray trajectories in the plane (w, z) (after [16]) (color online)

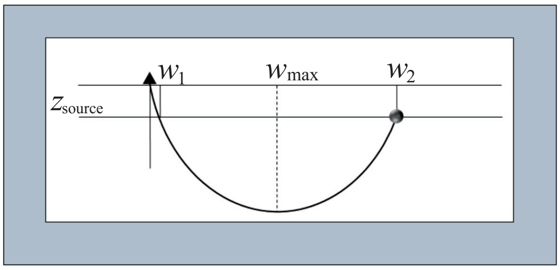

Fig. 6. Illustration of the testing data

Table 2

Illustration of the modeling error for each ray from Table 1

Station name	Depth, km	Error (radians)
Hau	4.4	0.005754648
Hei	3.7	0.004890067
Hau	1.9	0.061486027
Kro	1.5	0.063026168
Kro	2.9	0.023682692

Table 2 shows the difference between the values of the incident angle for the one-gradient model using only the formula of Euler's method (after [16]) and the values calculated for the same parameters of the one-gradient model applying the technique merging Euler's method with trigonometric formulas by reducing the derived two-gradient model to a one-gradient case. One can see that the modeling error is within the limits of accuracy. Thus, the correctness of the formula (18) has been verified for each ray.

4. Method for determining a formula to calculate the angle of incidence at the receiver

The method to find the formula for the angle of incidence on the receiver in a two-gradient medium includes the mathematical calculations (see Subsections 2.1–2.5), considering the physics of wave propagation. In the first-gradient medium, equation (7) is derived linking the angle of incidence on the receiver and the angle of incidence on the boundary of two gradients. For this, the property of ray reversibility is used. The tangent function is determined for each angle through the parameters describing the velocity growth with depth in the first-gradient medium. The relationship between the tangents of the angles is established by dividing one equation by another.

According to Snell's law, we obtain a formula for the known trigonometric relationship between the sine function of the angle of incidence at the boundary depth and the sine function of the angle of refraction through the parameters of the first and second gradients. The tangent of the angle of refraction is determined through the parameters of the medium of the second gradient by an algorithm that is the same as for the angles in the medium of the first-gradient (Subsection 2.3). Applying the known trigonometric equation relating the sine function and the inverse tangent function, we can obtain a formula for calculating the square of the sine of the refraction angle. Thus, the square sine of the refraction angle is defined twice through Snell's law and by utilizing a trigonometric equation. Equating the corresponding parts, we determine a formula for calculating the square sine of the angle of incidence on the depth, which is the boundary between two gradients. The parameters of both the first and second gradients are involved in this formula.

Again, applying the trigonometric equation relating the sine function and the inverse tangent function for the angle of incidence at the boundary depth, we derive a formula for determining the square tangent of the angle of incidence on the receiver.

Thus, the strategy of the method is as follows. It is necessary to find the "crossing" angle, which will be determined twice. From one direction, it will be related to the angle of incidence on the receiver for the ray path in the medium of the first-gradient, and from the other direction, it will be expressed through the parameters describing the medium of the second gradient. A simple substitution of the formula obtained for the "crossing" angle from the second direction into the expression for the first direction will give the desired formula for the angle of incidence on the receiver.

The equation of the basic angle of the ray path was tested through the transition from parameters for the two-gradient model to the parameters for the one-gradient model. An independent validation was performed by examining the difference between angle values known for a set of experimental data (published a few years ago in [16]) and the basic angle values calculated by applying the derived equation. The correctness of the result can be verified by direct substitution of the numerical data given in Table 1 (after [16]) into the formula (20).

Note that the basic formula (18) we derived for the incident angle on the receiver contains arithmetic operations and the trigonometric square function. This can lead to insignificant computing errors because numbers participate in calculations with rounding-off errors. It is known that for iterative approaches, including the Runge – Kutta method, this error can be critical and affect the correctness of the solution.

5. Discussion of application fields of the obtained solution

In this paper, we have obtained formulas for determining key angles through the parameters of the first and second gradients in a two-gradient model: the angle of incidence on the depth of the boundary between gradients (equation (16)), the angle of refraction (equation (15)), and finally the angle of incidence at the receiver (equation (18)). These important angles define the ray trace. The angle of incidence on the receiver is the main parameter for calculating the theoretical travel time. In seismic tomography, the efficiency of solutions to the direct and inverse problems depends on the difference between the theoretical travel time values and experimental observations.

Potential areas for the utilization of formulas constructed in the paper are beyond the scope of seismic tomography. Possible applications are in acoustic, optics, meteorology, astronomy, etc.

Let us consider the ray acoustic tasks, in which the sound velocity is not constant, and the rays are curved. For instance, the author [17] analyzed the sound propagation in the ocean. The model was considered for the medium with the sound velocity varying with depth. The ray angle was used to find a formula for the ray's radius of curvature at a defined depth. The main parameter of ray tracing is the initial ray angle at the source. The important contribution of the author [17] is the technique for underwater acoustic propagation over a layered bottom with a thin fluid sedimentary layer under the condition of moderately range-varying bathymetry.

Another application can be performed for light propagation theory, when the curved rays arise under the influence of gravity centers (lenses, galaxy) in the atmosphere. Coordinates of the ray path can be found by solving the system of differential equations with Euler's and Runge – Kutta iterative methods, not analytically [18]. The increase in the distance between the gravity center and the light particles leads to a decrease in the attenuation function and a change in the radii of the curved rays.

The author [18] discussed the intersection of curved light rays with massive gravity objects (black holes). The effects of shading can be calculated. However, shading models take into account the angle of incidence. A problem arises with the determination of this angle. Note that our research showed the incidence angle of the ray at any point can be calculated using the property of a ray in reverse.

Conclusion

The paper presents the technology established for determining the precise path of a ray in a two-gradient medium, given the starting positions of the receivers and sources, under the boundary condition of a gradient change at a certain depth. The equation derived for the basic angle in the two-gradient model can be applied to a specific case of a one-gradient model. The method involves a sequence of steps to solve the underdeterminacy problem in the ray tracing. It was found that the number of unknown parameters can be decreased if the property of the ray inverse trajectory is used. Numerous tests have shown that the modeling error is significantly reduced by using the coordinate systems, each of which is connected to the medium of one separate gradient. Trigonometric manipulations of Euler's method solutions for the different systems provide the formulas to calculate an accurate ray path in a two-gradient medium.

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Поступила в редакцию / Received 30.09.2025
Принята к публикации / Accepted 06.03.2026
Опубликована / Published 01.06.2026