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Article

On the convergence of the order-preserving weak greedy algorithm for subspaces generated by the Szegö kernel in the Hardy space

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Abstract. In this article we consider representing properties of subspaces generated by the Szegö kernel. We examine under which conditions on the sequence of points of the unit disk the order-preserving weak greedy algorithm for appropriate subspaces generated by the Szegö kernel converges. Previously, we constructed a representing system based on discretized Szegö kernels. The aim of this paper is to find an effective algorithm to get such representation, and we draw on the work of Silnichenko that introduced the notion of the order-preserving weak greedy algorithm. By selecting a special sequence of discretization points we refine one of Totik's results on the approximation of functions in the Hardy space using Szegö kernels. As the main result we prove the convergence criteria of the order-preserving weak greedy algorithm for subspaces generated by the Szegö kernel in the Hardy space.

Keywords: representing systems, Szegö kernel, order-preserving weak greedy algorithm
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О сходимости порядкосохраняющего слабого жадного алгоритма для подпространств, порожденных ядром Сегё в пространстве Харди

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Аннотация. В данной статье рассматриваются представляющие свойства подпространств, порожденных ядром Сегё. Дается ответ на следующий вопрос: при каких условиях на последовательность точек единичного диска имеет место сходимость порядкосохраняющего слабого жадного алгоритма для соответствующих подпространств, порожденных ядром Сегё. Ранее нами было показано существование системы представления на основе дискретизированных ядер Сегё. В данной работе мы переходим к эффективному алгоритму получения подобного представления и опираемся на работу Сильниченко, в которой было введено понятие порядкосохраняющего слабого жадного алгоритма. Уточняется один результат Тотика о приближении функций из пространства Харди посредством ядер Сегё за счет выбора последовательности точек дискретизации специального вида. Как основной результат статьи доказывается критерий сходимости порядкосохраняющего слабого жадного алгоритма для подпространств, порожденных ядром Сегё в пространстве Харди.

Ключевые слова: системы представления, ядро Сегё, порядкосохраняющий слабый жадный алгоритм

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Introduction

Approximating and representing properties of reproducing kernels play an important role in many questions such as geometry of function spaces, distribution of zeros, multiplicative structures. For more details we refer the reader to [1–3]. We begin by giving several relevant definitions.

Definition 1 (Representing system)**.** A sequence $\{\varphi_n\}_{n=1}^{\infty}$ of non-zero vectors from the Banach space F is called *a representing system* if for each element $f \in F$ there exists a sequence of coefficients $\{c_n(f)\}_{n=1}^{\infty}$ such that a series

$$f = \sum_{n=1}^{\infty} c_n(f)\varphi_n \tag{1}$$

converges to f in norm of the space F

$$\lim_{n\to\infty} \left\| f - \sum_{k=1}^{n} c_k(f)\varphi_k \right\|_F = 0.$$

In this paper we consider representing properties of discretized reproducing kernels in the Hardy space H^2.

Definition 2 (Hardy space)**.** A Hardy space H^2 is a Hilbert space of analytic functions $f(z)$ on a unit disk $\mathbb{D} = \{z : |z| < 1\}$ for which the following integral is bounded when $r \to 1$

$$\|f\|_{H^2} = \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{it}) \right|^2 \mathrm{d}t \right)^{\frac{1}{2}} < \infty. \tag{2}$$

Definition 3 (Szegö kernel)**.** A reproducing kernel K of the Hardy space $H^2(\mathbb{D})$ is a function of two variables and is called the Szegö kernel (see for instance [4])

$$K(z,\zeta) = K_\zeta(z) = \frac{1}{1-\bar{\zeta}z} = \sum_{n=0}^{\infty} \overline{\zeta^n} z^n, \quad z,\zeta \in \mathbb{D}. \tag{3}$$

The Szegö kernel K_ζ discretized at point $\zeta \in \mathbb{D}$ is a function in H^2.

There was an open question posed in [5] whether there exists a representing system in H^2 space based on a sequence of discretized reproducing kernels $\{K_{\zeta_n}\}_{n=1}^{\infty}$. In our previous paper [6], we gave an affirmative answer to this question using a theory of Banach frames [7] which provides the most general representation and presented a way to select the kernel discretization points (see also [8]). The natural question arises whether it is possible to devise an algorithm to calculate the coefficients $c_n(f)$ of the representation (1). This work was intended as an attempt to answer this question using the theory of greedy algorithms.

Greedy algorithms are a popular method to find representations given that at each step we select a suboptimal solution. For a recent account of the theory we refer the reader to [9]. There are a lot of different types of greedy algorithms: the pure greedy algorithm, the orthogonal greedy algorithm, the weak greedy algorithm, the relaxed greedy algorithm, etc.

We are concerned only with the order-preserving weak greedy algorithm. This type of greedy algorithms preserves the order of elements in the representation and the convergence properties of this algorithm were considered by Silnichenko [10].

In his original work [10], Silnichenko considered quasi-normed spaces. A quasi-normed space X is a space such that for each $x, y \in X$ the inequality holds $\|x+y\| \leqslant C(\|x\|+\|y\|)$, where C is a constant.

Definition 4 (Space with a uniformly continuous quasi-norm)**.** A vector space X is a space with a uniformly continuous quasi-norm if for any $\epsilon > 0$, $R > 0$ there exists $\delta > 0$ such that for arbitrary $x \in X$ and $y \in X$ with $\|x\| < R$, $\|y\| < \delta$ the inequality holds $\|x+y\| \leqslant \|x\| + \epsilon$.

Definition 5 (Order-preserving weak greedy algorithm [10])**.** Let $\{L_k\}$ be a sequence of linear subspaces of X and $\{\alpha_k\}$ be a sequence of non-negative numbers. The order-preserving weak greedy algorithm for an arbitrary element $f \in X$ is defined using the following iterative procedure. Let the initial approximation be equal to zero $s_0(f) = 0$, the remainder $r_0(f) = f$ and the optimal approximating element $k_0(f) = 0$. If $s_n(f), r_n(f)$ and $k_n(f)$ are defined, then we can select $k_{n+1} = k_{n+1}(f) > k_n(f)$ and $\phi_{k_{n+1}} \in L_{k_{n+1}}$ such that

$$\|r_n(f) - \phi_{k_{n+1}}\| \leqslant \inf_{\phi \in L_k, k > k_n} \|r_n(f) - \phi\| + \alpha_n,$$

where α_n can be informally called a relaxation. Then let $s_{n+1}(f) = s_n(f) + \phi_{k_{n+1}}$ and $r_{n+1}(f) = f - s_{n+1}(f)$.

If for all $f \in X$ the term $r_n(f)$ converges to zero as $n \to \infty$ or equivalently, we have a representation $f = \sum_{n=1}^{\infty} \phi_{k_n}$, then it is said that the algorithm converges (i. e. $s_n = \sum_{\nu=1}^{n} \phi_{k_\nu}$). Our main result relies heavily on the following theorem of Silnichenko.

Theorem 1 (Convergence of order-preserving weak greedy algorithms [10]). *Let X be a space with a uniformly continuous quasi-norm. The order-preserving weak greedy algorithm using the system $\{L_k\}$ converges for an arbitrary sequence $\{\alpha_k\}_{k=0}^{\infty}$, $\alpha_k \to 0$ when $k \to \infty$ if and only if there exists $\sigma < 1$ such that for all $f \in X$ and N there are $n > N$ and $\phi \in L_n$ as*

$$\|f - \phi\| \leqslant \sigma \|f\|. \tag{4}$$

Results of Silnichenko are valid for arbitrary quasi-Banach spaces, but in the present work we consider only the Hilbert space H^2. As subspaces $L_k, k = 1, 2, \ldots$ we select spaces generated by the Szegö kernel $K_{\zeta_{k,j}}$, $j = 0, \ldots, n_k - 1$, discretized at suitable points $\zeta_{k,j} \in \mathbb{D}$:

$$L_k := \left[K_{\zeta_{k,j}}\right]_{j=0}^{n_k-1} = \operatorname{span}\left\{K_{\zeta_{k,j}}\right\}_{j=0}^{n_k-1}, \quad k = 1, 2, \ldots.$$

1. Main result

We select discretization points of the Szegö kernel in the same way we did in our work [6]

$$\zeta_{k,j} = r_k e^{-\frac{2\pi i j}{n_k}}, \quad j = 0, 1, \ldots, n_k - 1, \tag{5}$$

where we assume that

$$r_k \nearrow 1, \quad n_k \nearrow \infty \quad \text{as} \quad k \to \infty. \tag{6}$$

The main result of the paper can be formulated as the following theorem.

Theorem 2. *Let the sequence of subspaces $\{L_k\}$ be generated by the Szegö kernel discretized at points that satisfy (5) and (6). Then the order-preserving weak greedy algorithm converges in the space H^2 if and only if r_k and n_k satisfy the limit inequality*

$$\limsup_{k\to\infty} n_k(1 - r_k) > 0. \tag{7}$$

Informally speaking, in theorem 2 each subspace L_k is generated by the Szegö kernel discretized at n_k roots of unity placed on the radius r_k.

2. Proof

We start proving the main result with mentioning some useful definitions.

Definition 6 (Blaschke condition). A sequence of points $\{z_n\}_{n=0}^{\infty} \subset \mathbb{D}$ satisfies the Blaschke condition when

$$\sum_{n=0}^{\infty}(1 - |z_n|) < \infty. \tag{8}$$

Definition 7 (Blaschke product). A Blaschke product is a function of the form

$$B(z) = \prod_{n\geqslant 1} \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \overline{z_n} z}, \tag{9}$$

where $\{z_1, z_2, \ldots\}$ are points that satisfy the Blaschke condition.

The Blaschke product can be finite or infinite and in the current work we use only the former one. It is clear from the definition of the Blashcke product that it has zeros at points $\{z_1, z_2, \ldots\}$.

Definition 8 (Distance)**.** By the distance between a function $f \in H^2$ and a linear subspace $L_k \subset H^2$ defined as a (closed) linear span $\left[K_{\zeta_{k,j}}\right]_{j=0}^{n_k-1}$ of discretized kernels $\left\{K_{\zeta_{k,j}}\right\}_{j=0}^{n_k-1}$ we mean

$$\operatorname{dist}(f,\ L_k) = \operatorname{dist}\left(f,\ \left[K_{\zeta_{k,j}}\right]_{j=0}^{n_k-1}\right) = \min_{a_0,\dots a_{n_k-1}} \left\| f - \sum_{j=0}^{n_k-1} a_j K_{\zeta_{k,j}} \right\|.$$

The following lemma is of crucial significance.

Lemma 1. *Let $0 < r < 1$. The distance from an arbitrary polynomial $p_n = \sum_{j=0}^{n-1} a_j z^j$ of degree $< n$ to the linear space defined by the system of Szegö kernels $\{K_j\}_{j=0}^{n-1}$ discretized at points*

$$z_j = re^{-\frac{2\pi i j}{n}}, \quad j = 0, 1, \dots, n-1, \tag{10}$$

satisfies the following inequality

$$\operatorname{dist}\left(p_n, [K_j]_{j=0}^{n-1}\right) \leqslant r^n \|p_n\|_{H^2}. \tag{11}$$

Proof. By corollary from the Hahn – Banach theorem, in an arbitrary Banach space B the distance from a function $f \in B$ to the subspace $L \subset B$ can be expressed as

$$\operatorname{dist}(f, L) = \sup_{\|g\|=1,\ g \in L^{\perp}} |\langle f, g\rangle|. \tag{12}$$

Let $\mathscr{K}_{r,n}$ be an n-dimensional space generated by Szegö kernels K_j discretized at points $z_j = re^{-\frac{2\pi i j}{n}}$, $j = 0, 1, \dots n-1$, then the orthogonal complement of $\mathscr{K}_{r,n}$ [3]

$$\mathscr{K}_{r,n}^{\perp} = \left\{ f \in H^2 : f\left(re^{-\frac{2\pi i j}{n}}\right) = 0,\ j = 0, \dots n-1 \right\}, \tag{13}$$

since $f(z) = 0$ if and only if $\langle f, K_j\rangle = 0$, $j = 0, \dots, n-1$.

If an analytic function $f \in H^2$ has zero values at points $z_1, z_2, \dots$ which satisfy the Blaschke condition (8), then $f = Bh$, where B is the corresponding Blashcke product and $h \in H^2$, and the converse is also true. Therefore we can rewrite (13) as

$$\mathscr{K}_{r,n}^{\perp} = BH^2 = \{Bh : h \in H^2\},$$

where $B = B_{r,n}$ is a Blaschke product with zeros $z_j = re^{-\frac{2\pi i j}{n}}$, $j = 0, 1, \dots n-1$.

Using (12) we can find the distance between the polynomial p_n and the n-dimensional space generated by the Szegö kernel $\mathscr{K}_{r,n}$

$$\operatorname{dist}(p_n, \mathscr{K}_{r,n}) = \sup_{\|g\|=1,\ g \in \mathscr{K}_{r,n}^{\perp}} |\langle p_n, g\rangle| = \sup_{\|h\|=1} |\langle p_n, B_{r,n} h\rangle|. \tag{14}$$

Given that discretization points have a special form (10) we can rewrite the definition of the Blaschke product (9) for our case as

$$B_{r,n}(z) = \prod_{j=0}^{n-1} \frac{re^{-\frac{2\pi i j}{n}} - z}{1 - zre^{\frac{2\pi i j}{n}}} = \frac{r^n - z^n}{1 - r^n z^n},$$

where in the numerator there are n-roots of unity on the radius r and in the denominator there are n-roots of unity on the radius $1/r$.

Note that we can repsesent the Blaschke product as a series having a lacunary structure

$$B_{r,n} = \frac{r^n - z^n}{1 - r^n z^n} = \sum_{j=0}^{\infty} b_j z^{nj}, \tag{15}$$

where $b_0 = r^n$ and $b_j = \left(r^n - \frac{1}{r^n}\right) r^{nj}$.

Now let $h(z) = \sum_{j=0}^{\infty} c_j z^j$, $\|h\| = 1$ and taking into account (15) we can write the product $B_{r,n}(z)h(z)$ in the following way

$$B_{r,n}(z)h(z) = r^n \sum_{j=0}^{n-1} c_j z^j + \ldots, \tag{16}$$

since only these terms are present for z^j, $j < n$.

By the conditions of the theorem $p_n = \sum_{j=0}^{n-1} a_j z^j$. On substituting p_n and (16) into (14) we obtain

$$|\langle p_n, B_{r,n} h\rangle| = r^n \left| \sum_{j=0}^{n-1} a_j \overline{c_j} \right| \leqslant r^n \|p_n\| \|h\| = r^n \|p_n\|,$$

where we used the fact that the maximum degree of p_n is $n-1$, the definition of a scalar product using Taylor expansion coefficients and a Cauchy – Shwartz inequality. □

Remark 1. Totik proved in [11] that for arbitrary selection of points $\{\zeta_{k,j}\}_{j=0}^{n_k-1} \subset \mathbb{D}$

$$\operatorname{dist}(1,\ L_k) = \operatorname{dist}\left(1,\ \left[K_{\zeta_{k,j}}\right]_{j=0}^{n_k-1}\right) = \prod_{j=0}^{n_k-1} |\zeta_{k,j}|. \tag{17}$$

Yet in lemma 1 we select discretization points in a special way (10) which allows us to get a bound on a distance from L_k to each polynomial p_{n_k}, $\deg p_{n_k} < n_k$, not only $p_{n_k} = 1$. Equality (17) shows that inequality (11) is exact.

Remark 2. Note that we can take sequences $\{r_k\}$, $\{n_k\}$ such that the sequence of best approximations of $f \in H^2$ by $\{L_k\}$ vanishes:

$$\lim_{k\to\infty} \operatorname{dist}(f, \left[K_{\zeta_{k,j}}\right]_{j=0}^{n_k-1}) = 0.$$

Indeed, it follows from lemma 1 if and only if $\lim_{k\to\infty} n_k(1 - r_k) = \infty$.

Now we are ready to prove the main theorem. To do so it is enough to check that condition (4) is true, which is equivalent to

$$\liminf \operatorname{dist}(f, L_k) < \sigma\|f\|, \quad f \in H^2. \tag{18}$$

Let $\epsilon > 0$ and p_n be a polynomial of degree $< n$ that approximates f

$$\|f - p_n\| \leqslant \epsilon\|f\|$$

and let k be large enough so $n_k > n$. Then using lemma 1 and a triangle inequality

$$\operatorname{dist}(f, L_k) = \operatorname{dist}(f - p_n, L_k) + \operatorname{dist}(p_n, L_k) \leqslant \epsilon\|f\| + r_k^{n_k}\|p_n\| \leqslant (r_k^{n_k} + r_k^{n_k}\epsilon + \epsilon)\|f\|. \quad (19)$$

Therefore the criteria of Silnichenko (4) is satisfied if

$$\liminf r_k^{n_k} < 1$$

which is equivalent to (7). So the order-preserving weak greedy algorithm converges if (7) holds. Given remark 1 (exactness of inequality (17)) one sees immediately that the converse is also true.

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