Научный отдел

МАТЕМАТИКА

Известия Саратовского университета. Новая серия. Серия: Математика. Механика. Информатика. 2022. Т. 22, вып. 1. С. 4–14
Izvestiya of Saratov University. Mathematics. Mechanics. Informatics, 2022, vol. 22, iss. 1, pp. 4–14
https://mmi.sgu.ru
https://doi.org/10.18500/1816-9791-2022-22-1-4-14

Article

New exact solutions for the two-dimensional Broadwell system

S. A. Dukhnovsky

Moscow State University of Civil Engineering, 26 Yaroslavskoe Shosse, Moscow 129337, Russia

Sergey A. Dukhnovsky, sergeidukhnvskijj@rambler.ru, https://orcid.org/0000-0001-9643-7394

Abstract. In this paper, we consider the discrete kinetic Broadwell system. This system is a nonlinear hyperbolic system of partial differential equations. The two-dimensional Broadwell system is the kinetic Boltzmann equation, and for this model momentum and energy are conserved. In the kinetic theory of gases, the system describes the motion of particles moving on a two-dimensional plane, the right-hand side is responsible for pair collisions of particles. For the first time, new traveling wave solutions are found using the $\exp(-\varphi(\xi))$-expansion method. This method is as follows. The solution is sought in the form of a traveling wave. In this case, the system is reduced to a system of ordinary differential equations. Further, the solution is sought according to this method in the form of an exponential polynomial, depending on an unknown function that satisfies a certain differential equation. Solutions of the differential equation themselves are known. The summation is carried out up to a certain positive number, which is determined by the balance between the highest linear and non-linear terms. Further, the proposed solution is substituted into the system of differential equations and coefficients at the same exponential powers are collected. Solving systems of algebraic equations, we find unknown coefficients and write the original solution. This method is universal and allows us to obtain a large number of solutions, namely, kink solutions, singular kink solutions, periodic solutions, and rational solutions. Corresponding graphs of some solutions are presented by the Mathematica package. With the help of computerized symbolic computation, we obtain new solutions. Similarly, it is possible to find exact solutions for other kinetic models.

Keywords: two-dimensional Broadwell system, traveling wave solutions, analytical method, kinetic Boltzmann equation, Knudsen parameter
For citation: Dukhnovsky S. A. New exact solutions for the two-dimensional Broadwell system. *Izvestiya of Saratov University. Mathematics. Mechanics. Informatics*, 2022, vol. 22, iss. 1, pp. 4–14. https://doi.org/10.18500/1816-9791-2022-22-1-4-14

Научная статья
УДК 517.951

Новые точные решения для двумерной системы Бродуэлла

С. А. Духновский

Национальный исследовательский Московский государственный строительный университет, Россия, 129337, г. Москва, ул. Ярославское шоссе, д. 26

Духновский Сергей Анатольевич, кандидат физико-математических наук, преподаватель кафедры прикладной математики, sergeidukhnvskijj@rambler.ru, https://orcid.org/0000-0001-9643-7394

Аннотация. В статье рассмотрена дискретная кинетическая система Бродуэлла. Данная система является нелинейной гиперболической системой уравнений в частных производных. Двумерная система Бродуэлла представляет собой кинетическое уравнение Больцмана, и для этой модели импульс и энергия сохраняются. В кинетической теории газов система описывает движение частиц на двумерной плоскости, при этом правая часть системы отвечает за парные столкновения частиц. Впервые новые решения бегущей волны найдены с использованием метода $\exp(-\varphi(\xi))$-разложения. Данные метод состоит в следующем. Решение ищется в виде бегущей волны. В этом случае система сводится к системе обыкновенных дифференциальных уравнений. Далее решение ищется согласно данному методу в виде полинома по экспонентам (сумма ряда), зависящего от неизвестной функции, которая удовлетворяет определенному дифференциальному уравнению. При этом известны сами решения дифференциального уравнения. Суммирование ведется до конкретного положительного числа, которое определяется посредством баланса между наивысшими линейными и нелинейными членами. Далее предполагаемое решение подставляется в систему дифференциальных уравнений и собираются коэффициенты при одинаковых степенях экспонент. Решая системы алгебраических уравнений, мы находим неизвестные коэффициенты и записываем исходное решение. Данный метод является универсальным и позволяет получить большое число решений, а именно кинковые, сингулярные кинковые, периодические и рациональные решения. Соответствующие графики некоторых решений представлены посредством пакета «Математика». С помощью компьютерных символьных вычислений получены новые решения. Аналогичным образом можно найти точные решения для других кинетических моделей.
Ключевые слова: двумерная система Бродуэлла, решения бегущей волны, аналитический метод, кинетическое уравнение Больцмана, параметр Кнудсена
Для цитирования: *Dukhnovsky S. A.* New exact solutions for the two-dimensional Broadwell system [*Духновский С. А.* Новые точные решения для двумерной системы Бродуэлла] // Известия Саратовского университета. Новая серия. Серия: Математика. Механика. Информатика. 2022. Т. 22, вып. 1. С. 4–14. https://doi.org/10.18500/1816-9791-2022-22-1-4-14

Introduction

Consider the two-dimensional Broadwell system [1,2]:

$$\begin{aligned}
&\partial_t u + \partial_x u = \frac{1}{\varepsilon}(wz - uv),\\
&\partial_t v - \partial_x v = \frac{1}{\varepsilon}(wz - uv), \quad x, y \in \mathbb{R}, \quad t > 0,\\
&\partial_t w + \partial_y w = \frac{1}{\varepsilon}(uv - wz),\\
&\partial_t z - \partial_y z = \frac{1}{\varepsilon}(uv - wz).
\end{aligned} \tag{1}$$

Here $u = u(x,y,t)$, $v = v(x,y,t)$, $w = w(x,y,t)$, $z = z(x,y,t)$ are the densities of four groups of particles with velocities $(1,0,0), (-1,0,0), (0,1,0), (0,-1,0)$, ε is the Knudsen parameter from the kinetic theory of gases. This system describes a rarefied gas consisting of four groups of particles. The interaction is as follows. The Broadwell system describes particles of four groups, namely, the first group of particles moves at a unit speed along with the axis Ox, and the second group moves at a unit speed in the opposite direction. The third and fourth groups move in a similar way. Particles of the first and second groups colliding cause a reaction that transfers into particles of the third and fourth groups. In turn, particles of the third and fourth groups transfer into particles of the first and second groups.

There are many methods for finding exact solutions such as the sine-cosine [3,4], the $\exp(-\varphi(\xi))$-function method [5], the G'/G-expansion method [6], the generalized G'/G-expansion method [7], the homogeneous balance method [8,9], the Riccati–Bernoulli sub-ODE method [10,11], the Jacobi elliptic function expansion method [12], the Exp-function method [13,14], the Kudryashov method [15], the first integral method [16] and others. The Broadwell system is a non-integrable system, i.e. the Painlevé test is not applicable. In [17], exact solutions of the Carleman system with conformable derivative were obtained via the generalized Bernoulli sub-ODE method. In [18,19], the authors found solutions of kinetic systems using the Bateman equation. Asymptotic stability of equilibrium states for the Carleman and Godunov–Sultangazin systems was proved in [20,21]. We, for the first time, will get solutions using the $\exp(-\varphi(\xi))$-expansion method.

1. Description of the $\exp(-\varphi(\xi))$-expansion method

Consider a given nonlinear equation

$$E(u, u_t, u_x, u_y, u_{xt}, u_{xy}, \ldots) = 0, \tag{2}$$

where $u = u(x,y,t)$ is an unknown function. We will look for a traveling wave solution $\xi = kx + ly + ct$, $u = U(\xi)$. Then (2) is reduced to the ordinary differential equation:

$$E(U, cU', kU', lU', klU'', klU'', \ldots) = 0. \tag{3}$$

According to the $\exp(-\varphi(\xi))$-expansion method, a solution is sought in the form

$$U(\xi) = \sum_{i=0}^{N} a_i \exp(-i\varphi(\xi)), \tag{4}$$

where $a_N \neq 0$ and $\varphi(\xi)$ satisfies the ODE in the following form

$$\varphi'(\xi) = \mu \exp(\varphi(\xi)) + \exp(-\varphi(\xi)) + \lambda\,. \tag{5}$$

The solutions of Eq. (5) have the following form:

1) when $\mu \neq 0,\ \lambda^2 - 4\mu > 0:$

$$\varphi(\xi) = \ln\left(\frac{-\sqrt{\lambda^2-4\mu}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(\xi+C)\right)-\lambda}{2\mu}\right), \tag{6}$$

and

$$\varphi(\xi) = \ln\left(\frac{-\sqrt{\lambda^2-4\mu}\coth\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(\xi+C)\right)-\lambda}{2\mu}\right); \tag{7}$$

2) when $\mu \neq 0,\ \lambda^2 - 4\mu < 0:$

$$\varphi(\xi) = \ln\left(\frac{\sqrt{4\mu-\lambda^2}\tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(\xi+C)\right)-\lambda}{2\mu}\right), \tag{8}$$

and

$$\varphi(\xi) = \ln\left(\frac{\sqrt{4\mu-\lambda^2}\cot\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(\xi+C)\right)-\lambda}{2\mu}\right); \tag{9}$$

3) when $\mu = 0,\ \lambda \neq 0,\ \lambda^2 - 4\mu > 0:$

$$\varphi(\xi) = -\ln\left(\frac{\lambda}{\exp(\lambda(\xi+C))-1}\right); \tag{10}$$

4) when $\mu \neq 0,\ \lambda \neq 0,\ \lambda^2 - 4\mu = 0:$

$$\varphi(\xi) = \ln\left(-\frac{2(\lambda(\xi+C)+2)}{\lambda^2(\xi+C)}\right); \tag{11}$$

5) when $\mu = 0,\ \lambda = 0,\ \lambda^2 - 4\mu = 0:$

$$\varphi(\xi) = \ln\left(\xi+C\right), \tag{12}$$

where C is an integrating constant.

The positive integer N is determined by a balance between the linear term of the highest order and the highest order nonlinear term. Substituting (4) into (3) and equating coefficients at $\exp(-i\varphi(\xi)), i = 0, 1, 2, \ldots$, to zero, we obtain a system of algebraic equations. Then we find the unknown coefficients and write our solutions.

2. Application of the $\exp(-\varphi(\xi))$-expansion method

We use the transformation

$$u = U(\xi), \quad v = V(\xi), \quad w = W(\xi), \quad z = Z(\xi), \quad \xi = kx + ly + ct.$$

Then the system has the form

$$\begin{aligned} U'(c+k) &= \frac{1}{\varepsilon}(WZ - UV), \\ V'(c-k) &= \frac{1}{\varepsilon}(WZ - UV), \\ W'(c+l) &= \frac{1}{\varepsilon}(UV - WZ), \\ Z'(c-l) &= \frac{1}{\varepsilon}(UV - WZ). \end{aligned} \tag{13}$$

We apply the following expansions by the $\exp(-\varphi(\xi))$-expansion method

$$\begin{aligned} U(\xi) &= \sum_{i=0}^{N} a_i \exp(-i\varphi(\xi)), \quad V(\xi) = \sum_{i=0}^{M} b_i \exp(-i\varphi(\xi)), \\ W(\xi) &= \sum_{i=0}^{P} c_i \exp(-i\varphi(\xi)), \quad Z(\xi) = \sum_{i=0}^{L} d_i \exp(-i\varphi(\xi)), \end{aligned} \tag{14}$$

where $N,\ M,\ P,\ L$ are positive integers.

Balancing between U' and UV yields $M = 1$. Similarly, we have

$$M + 1 = N + M, \quad P + 1 = P + L, \quad L + 1 = P + L,$$

so that $N = P = L = 1$. Then we seek the solution of (13) in the form

$$\begin{aligned} U &= a_0 + a_1 \exp(-\phi(\xi)), \quad V = b_0 + b_1 \exp(-\phi(\xi)), \\ W &= c_0 + c_1 \exp(-\phi(\xi)), \quad Z = d_0 + d_1 \exp(-\phi(\xi)). \end{aligned} \tag{15}$$

Substituting (15) into (13) and collecting coefficients of the order of $\exp(-i\varphi(\xi))$, $i = 0, 1, 2, \ldots$, we have

$$\begin{aligned} \frac{a_1 b_0}{\varepsilon} + \frac{a_0 b_1}{\varepsilon} - \frac{c_1 d_0}{\varepsilon} - \frac{c_0 d_1}{\varepsilon} - c\lambda a_1 - k\lambda a_1 &= 0, \\ \frac{a_1 b_1}{\varepsilon} - \frac{c_1 d_1}{\varepsilon} - c a_1 - k a_1 &= 0, \\ \frac{a_0 b_0}{\varepsilon} - \frac{c_0 d_0}{\varepsilon} - c\mu a_1 - k\mu a_1 &= 0, \end{aligned} \tag{16}$$

and

$$\begin{gathered}\frac{a_1b_0}{\varepsilon}+\frac{a_0b_1}{\varepsilon}-\frac{c_1d_0}{\varepsilon}-\frac{c_0d_1}{\varepsilon}-c\lambda b_1+k\lambda b_1=0,\\ \frac{a_1b_1}{\varepsilon}-\frac{c_1d_1}{\varepsilon}-cb_1+kb_1=0,\\ \frac{a_0b_0}{\varepsilon}-\frac{c_0d_0}{\varepsilon}-c\mu b_1+k\mu b_1=0,\end{gathered}\tag{17}$$

and

$$\begin{gathered}-\frac{a_1b_0}{\varepsilon}-\frac{a_0b_1}{\varepsilon}+\frac{c_1d_0}{\varepsilon}+\frac{c_0d_1}{\varepsilon}-c\lambda c_1-l\lambda c_1=0,\\ -\frac{a_1b_1}{\varepsilon}+\frac{c_1d_1}{\varepsilon}-cc_1-lc_1=0,\\ -\frac{a_0b_0}{\varepsilon}+\frac{c_0d_0}{\varepsilon}-c\mu c_1-l\mu c_1=0,\end{gathered}\tag{18}$$

and

$$\begin{gathered}-\frac{a_1b_0}{\varepsilon}-\frac{a_0b_1}{\varepsilon}+\frac{c_1d_0}{\varepsilon}+\frac{c_0d_1}{\varepsilon}-c\lambda d_1+l\lambda d_1=0,\\ -\frac{a_1b_1}{\varepsilon}+\frac{c_1d_1}{\varepsilon}-cd_1+ld_1=0,\\ -\frac{a_0b_0}{\varepsilon}+\frac{c_0d_0}{\varepsilon}-c\mu d_1+l\mu d_1=0.\end{gathered}\tag{19}$$

Solving together (16)–(19) by the Mathematica package, we obtain

$$\lambda=\frac{b_0(c^2-k^2)\Big(c(c_0+d_0)+(c_0-d_0)l\Big)(k^2-l^2)+b_0^2(c-k)(c^2-l^2)(k^2-l^2)}{b_0(c^2-k^2)(c^2-l^2)(k^2-l^2)\varepsilon}+$$
$$+\frac{(c+k)(c^2-l^2)\Big(c_0d_0(k^2-l^2)+(c^2-k^2)(c^2-l^2)\varepsilon^2\mu\Big)}{b_0(c^2-k^2)(c^2-l^2)(k^2-l^2)\varepsilon},$$

$$a_0=\frac{c_0d_0(k^2-l^2)+(c^2-k^2)(c^2-l^2)\varepsilon^2\mu}{b_0(k^2-l^2)},$$

$$a_1=\frac{(c-k)(c^2-l^2)\varepsilon}{k^2-l^2},\quad d_1=-\frac{(c^2-k^2)(c+l)\varepsilon}{k^2-l^2},$$

$$c_1=-\frac{(c^2-k^2)(c-l)\varepsilon}{k^2-l^2},\quad b_1=\frac{(c+k)(c^2-l^2)\varepsilon}{k^2-l^2},$$

$$b_0=b_0,\quad c_0=c_0,\quad d_0=d_0.$$

The exact solutions of the system (1) are:

1) when $\mu \neq 0,\ \lambda^2 - 4\mu > 0$,

$$u_1(\xi) = \frac{c_0 d_0(k^2 - l^2) + (c^2 - k^2)(c^2 - l^2)\varepsilon^2\mu}{b_0(k^2 - l^2)} - \\ - \frac{(c-k)(c^2-l^2)\varepsilon}{k^2 - l^2}\left(\frac{2\mu}{\sqrt{\lambda^2 - 4\mu}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(\xi + C)\right) + \lambda}\right), \quad (20)$$

$$v_1(\xi) = b_0 - \frac{(c+k)(c^2-l^2)\varepsilon}{k^2 - l^2}\left(\frac{2\mu}{\sqrt{\lambda^2 - 4\mu}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(\xi + C)\right) + \lambda}\right), \quad (21)$$

$$w_1(\xi) = c_0 + \frac{(c^2-k^2)(c-l)\varepsilon}{k^2 - l^2}\left(\frac{2\mu}{\sqrt{\lambda^2 - 4\mu}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(\xi + C)\right) + \lambda}\right), \quad (22)$$

$$z_1(\xi) = d_0 + \frac{(c^2-k^2)(c+l)\varepsilon}{k^2 - l^2}\left(\frac{2\mu}{\sqrt{\lambda^2 - 4\mu}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(\xi + C)\right) + \lambda}\right), \quad (23)$$

where $\xi = kx + ly + ct$ and C is an arbitrary constant;

2) when $\mu \neq 0,\ \lambda^2 - 4\mu < 0$,

$$u_2(\xi) = \frac{c_0 d_0(k^2 - l^2) + (c^2 - k^2)(c^2 - l^2)\varepsilon^2\mu}{b_0(k^2 - l^2)} + \\ + \frac{(c-k)(c^2-l^2)\varepsilon}{k^2 - l^2}\left(\frac{2\mu}{\sqrt{4\mu - \lambda^2}\tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(\xi + C)\right) - \lambda}\right), \quad (24)$$

$$v_2(\xi) = b_0 + \frac{(c+k)(c^2-l^2)\varepsilon}{k^2 - l^2}\left(\frac{2\mu}{\sqrt{4\mu - \lambda^2}\tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(\xi + C)\right) - \lambda}\right), \quad (25)$$

$$w_2(\xi) = c_0 - \frac{(c^2-k^2)(c-l)\varepsilon}{k^2 - l^2}\left(\frac{2\mu}{\sqrt{4\mu - \lambda^2}\tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(\xi + C)\right) - \lambda}\right), \quad (26)$$

$$z_2(\xi) = d_0 - \frac{(c^2-k^2)(c+l)\varepsilon}{k^2 - l^2}\left(\frac{2\mu}{\sqrt{4\mu - \lambda^2}\tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(\xi + C)\right) - \lambda}\right), \quad (27)$$

where $\xi = kx + ly + ct$ and C is an arbitrary constant;

3) when $\mu = 0,\ \lambda \neq 0,\ \lambda^2 - 4\mu > 0$,

$$u_3(\xi) = \frac{c_0 d_0(k^2 - l^2) + (c^2 - k^2)(c^2 - l^2)\varepsilon^2\mu}{b_0(k^2 - l^2)} + \frac{(c-k)(c^2-l^2)\varepsilon}{k^2 - l^2}\left(\frac{\lambda}{\exp(\lambda(\xi + C)) - 1}\right), \quad (28)$$

$$v_3(\xi) = b_0 + \frac{(c+k)(c^2-l^2)\varepsilon}{k^2 - l^2}\left(\frac{\lambda}{\exp(\lambda(\xi + C)) - 1}\right), \quad (29)$$

$$w_3(\xi) = c_0 - \frac{(c^2-k^2)(c-l)\varepsilon}{k^2 - l^2}\left(\frac{\lambda}{\exp(\lambda(\xi + C)) - 1}\right), \quad (30)$$

$$z_4(\xi) = d_0 - \frac{(c^2-k^2)(c+l)\varepsilon}{k^2 - l^2}\left(\frac{\lambda}{\exp(\lambda(\xi + C)) - 1}\right), \quad (31)$$

where $\xi = kx + ly + ct$ and C is an arbitrary constant;

4) when $\mu \neq 0,\ \lambda \neq 0,\ \lambda^2 - 4\mu = 0$,

$$u_4(\xi) = \frac{c_0 d_0(k^2-l^2)+(c^2-k^2)(c^2-l^2)\varepsilon^2\mu}{b_0(k^2-l^2)} - \frac{(c-k)(c^2-l^2)\varepsilon}{k^2-l^2}\left(\frac{\lambda^2(\xi+C)}{2(\lambda(\xi+C)+2)}\right), \tag{32}$$

$$v_4(\xi) = b_0 - \frac{(c+k)(c^2-l^2)\varepsilon}{k^2-l^2}\left(\frac{\lambda^2(\xi+C)}{2(\lambda(\xi+C)+2)}\right), \tag{33}$$

$$w_4(\xi) = c_0 + \frac{(c^2-k^2)(c-l)\varepsilon}{k^2-l^2}\left(\frac{\lambda^2(\xi+C)}{2(\lambda(\xi+C)+2)}\right), \tag{34}$$

$$z_4(\xi) = d_0 + \frac{(c^2-k^2)(c+l)\varepsilon}{k^2-l^2}\left(\frac{\lambda^2(\xi+C)}{2(\lambda(\xi+C)+2)}\right), \tag{35}$$

where $\xi = kx + ly + ct$ and C is an arbitrary constant;

5) when $\mu = 0,\ \lambda = 0,\ \lambda^2 - 4\mu = 0$,

$$u_5(\xi) = \frac{c_0 d_0(k^2-l^2)+(c^2-k^2)(c^2-l^2)\varepsilon^2\mu}{b_0(k^2-l^2)} + \frac{(c-k)(c^2-l^2)\varepsilon}{k^2-l^2}\left(\frac{1}{\xi+C}\right), \tag{36}$$

$$v_5(\xi) = b_0 + \frac{(c+k)(c^2-l^2)\varepsilon}{k^2-l^2}\left(\frac{1}{\xi+C}\right), \tag{37}$$

$$w_5(\xi) = c_0 - \frac{(c^2-k^2)(c-l)\varepsilon}{k^2-l^2}\left(\frac{1}{\xi+C}\right), \tag{38}$$

$$z_5(\xi) = d_0 - \frac{(c^2-k^2)(c+l)\varepsilon}{k^2-l^2}\left(\frac{1}{\xi+C}\right), \tag{39}$$

where $\xi = kx + ly + ct$ and C is an arbitrary constant.

3. Graphs of the obtained solutions

Here we will plot some graphs of the obtained solutions. Equation (20) represents the kink wave. Figure 1 shows the exact 3D kink-type solution of equation (20) for $b_0 = 1$, $\varepsilon = 1$, $c = 1$, $k = 2$, $l = 3$, $c_0 = 1$, $d_0 = 1$, $\mu = 1$, $C = 1$ and $-10 \leqslant x \leqslant 10$, $0 \leqslant t \leqslant 10$.

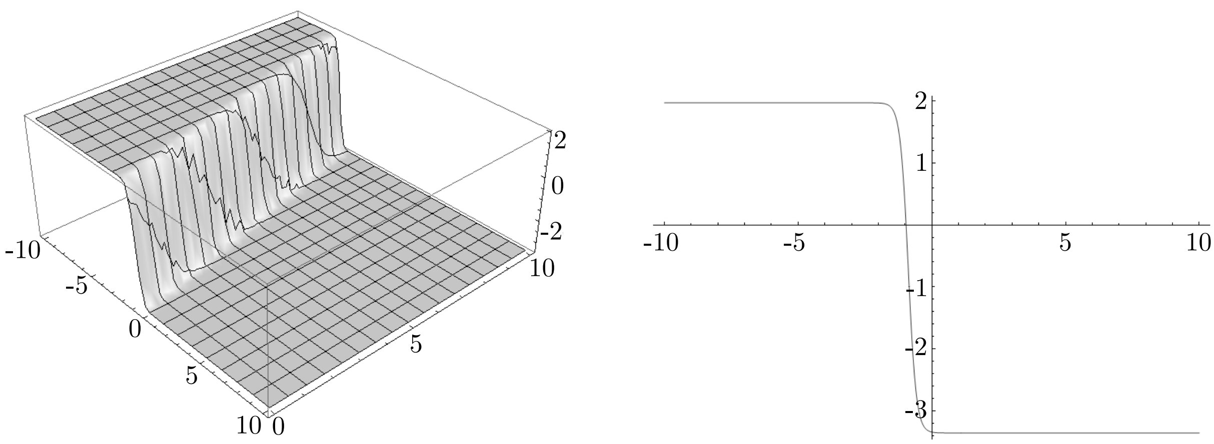

Fig. 1. Kink solution of $u_1(\xi)$ for $b_0 = 1$, $\varepsilon = 1$, $c = 1$, $k = 2$, $l = 3$, $c_0 = 1$, $d_0 = 1$, $\mu = 1$, $C = 1$. The left figure shows the 3-D plot and the right figure shows the 2-D plot for $t = 0$

Equation (24) represents the periodic traveling wave solution. Figure 2 shows the exact 3D periodic solution of equation (24) for $b_0 = 3$, $\varepsilon = 1$, $c = 1$, $k = 2$, $l = 3$, $c_0 = 2$, $d_0 = 1$, $\mu = 1$, $C = 1$ and $-10 \leqslant x \leqslant 10$, $0 \leqslant t \leqslant 10$.

Equation (28) represents the singular kink solution. Figure 3 shows the exact 3D singular kink solution of equation (28) for $b_0 = -1$, $\varepsilon = 1$, $c = -1$, $k = 3$, $l = 2$, $c_0 = 3$, $d_0 = 1$, $\mu = 0$, $C = 1$ and $-10 \leqslant x \leqslant 10$, $0 \leqslant t \leqslant 10$.

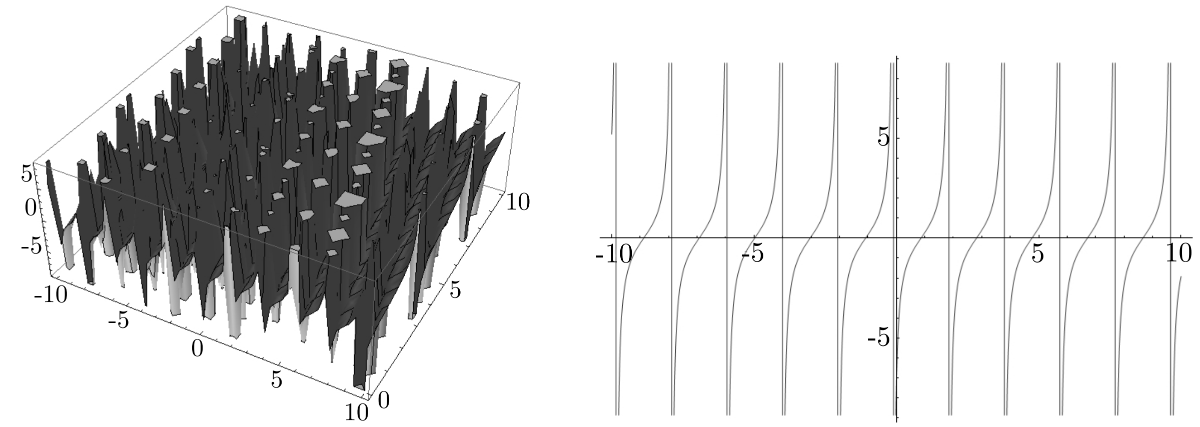

Fig. 2. Periodic wave solution of $u_2(\xi)$ for $b_0 = 3$, $\varepsilon = 1$, $c = 1$, $k = 2$, $l = 3$, $c_0 = 2$, $d_0 = 1$, $\mu = 1$, $C = 1$. The left figure shows the 3-D plot and the right figure shows the 2-D plot for $t = 0$

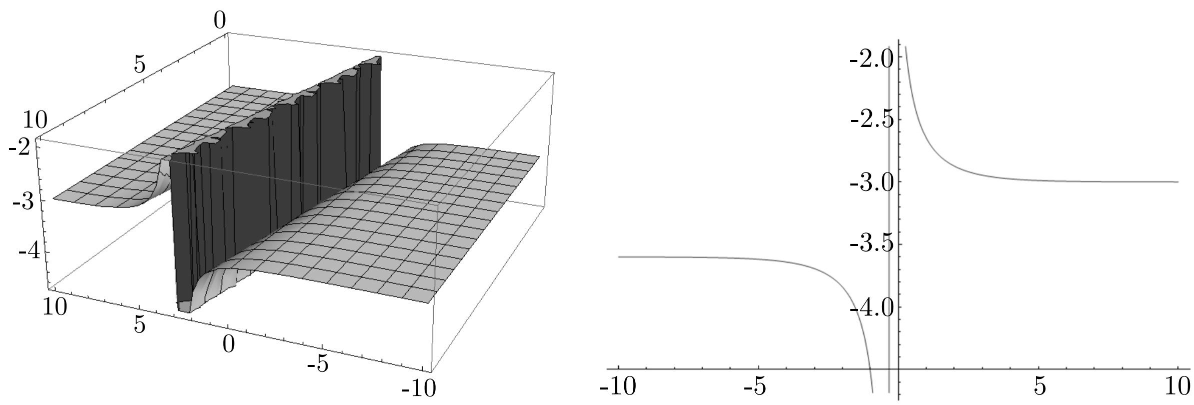

Fig. 3. Singular kink solution of $u_3(\xi)$ for $b_0 = -1$, $\varepsilon = 1$, $c = -1$, $k = 3$, $l = 2$, $c_0 = 3$, $d_0 = 1$, $\mu = 0$, $C = 1$. The left figure shows the 3-D plot and the right figure shows the 2-D plot for $t = 0$

Conclusion

In this work, we have found the exact traveling wave solutions of the kinetic Broadwell system by using the $\exp(-\varphi(\xi))$-expansion method. All of the above solutions have been verified using the Mathematica package. In the future, the solutions of the remaining kinetic models will be found.

References

1. Radkevich E. V. On the large-time behavior of solutions to the Cauchy problem for a 2-dimensional discrete kinetic equation. *Journal of Mathematical Sciences*, 2014, vol. 202, no. 5, pp. 735–768. https://doi.org/10.1007/s10958-014-2074-x

2. Godunov S. K., Sultangazin U. M. On discrete models of the kinetic Boltzmann equation. *Russian Mathematical Surveys*, 1971, vol. 26, no. 3, pp. 1–56. https://doi.org/10.1070/RM1971v026n03ABEH003822
3. Wazwaz A. M. A sine-cosine method for handing nonlinear wave equations. *Mathematical and Computer Modelling*, 2004, vol. 40, iss. 5–6, pp. 499–508. https://doi.org/10.1016/j.mcm.2003.12.010
4. Wazwaz A. M. The tanh and the sine-cosine methods for the complex modified KdV and the generalized KdV equations. *Computers and Mathematics with Applications*, 2005, vol. 49, iss. 7–8, pp. 1101–1112. https://doi.org/10.1016/j.camwa.2004.08.013
5. Alam M. N., Alam M. M. An analytical method for solving exact solutions of a nonlinear evolution equation describing the dynamics of ionic currents along microtubules. *Journal of Taibah University for Science*, 2017, vol. 11, iss. 6, pp. 939–948. https://doi.org/10.1016/j.jtusci.2016.11.004
6. Jafari H., Kadkhoda N., Biswas A. The G'/G-expansion method for solutions of evolution equations from isothermal magnetostatic atmospheres. *Journal of King Saud University — Science*, 2013, vol. 25, iss. 1, pp. 57–62. https://doi.org/10.1016/j.jksus.2012.02.002
7. Alam M. N., Akbar M. A., Mohyud-Din S. T. General traveling wave solutions of the strain wave equation in microstructured solids via the new approach of generalized G'/G-expansion method. *Alexandria Engineering Journal*, 2014, vol. 53, iss. 1, pp. 233–241. https://doi.org/10.1016/j.aej.2014.01.002
8. Fan E., Zhang H. A note on the homogeneous balance method. *Physics Letters A*, 1998, vol. 246, iss. 5, pp. 403–406. https://doi.org/10.1016/S0375-9601(98)00547-7
9. Bai C. L. Extended homogeneous balance method and Lax pairs, Bäcklund transformation. *Communications in Theoretical Physics*, 2002, vol. 37, no. 6, pp. 645–648. https://doi.org/10.1088/0253-6102/37/6/645
10. Alharbi A. R., Almatrafi M. B. New exact and numerical solutions with their stability for Ito integro-differential equation via Riccati–Bernoulli sub-ODE method. *Journal of Taibah University for Science*, 2020, vol. 14, iss. 1, pp. 1447–1456. https://doi.org/10.1080/16583655.2020.1827853
11. Yang X. F., Deng Z. C., Wei Y. A Riccati–Bernoulli sub-ODE method for nonlinear partial differential equations and its application. *Advances in Continuous and Discrete Models*, 2015, vol. 2015, no. 117, pp. 1–17. https://doi.org/10.1186/s13662-015-0452-4
12. Alquran M., Jarrah A. Jacobi elliptic function solutions for a two-mode KdV equation. *Journal of King Saud University — Science*, 2019, vol. 31, iss. 4, pp. 485–489. https://doi.org/10.1016/j.jksus.2017.06.010
13. Zhang W. The extended tanh method and the exp-function method to solve a kind of nonlinear heat equation. *Mathematical Problems in Engineering*, vol. 2010, Article ID 935873, 12 p. https://doi.org/10.1155/2010/935873
14. He J. H., Wu X. H. Exp-function method for nonlinear wave equations. *Chaos, Solitons & Fractals*, 2006, vol. 30, iss. 3, pp. 700–708. https://doi.org/10.1016/j.chaos.2006.03.020
15. Gaber A. A., Aljohani A. F., Ebaid A., Tenreiro Machado J. The generalized Kudryashov method for nonlinear space-time fractional partial differential equations of Burgers type. *Nonlinear Dynamics*, 2019, vol. 95, no. 3, pp. 361–368. https://doi.org/10.1007/s11071-018-4568-4
16. Nizovtseva I. G., Galenko P. K., Alexandrov D. V., Vikharev S. V., Titova E. A., Sukhachev I. S. Traveling waves in a profile of phase field: exact analytical solutions of a hyperbolic Allen–Cahn equation. *Vestnik Udmurtskogo Universiteta. Matematika. Mekhanika. Komp'yuternye Nauki*, 2016, vol. 26, no. 2, pp. 245–257 (in Russian). https://doi.org/10.20537/vm160211

17. Tchier F., Inc M., Yusuf A. Symmetry analysis, exact solutions and numerical approximations for the space-time Carleman equation in nonlinear dynamical systems. *The European Physical Journal Plus*, 2019, vol. 134, no. 250, pp. 1–18. https://doi.org/10.1140/epjp/i2019-12586-1
18. Dukhnovskii S. A. Solutions of the Carleman system via the Painlevé expansion. *Vladikavkaz Mathematical Journal*, 2020, vol. 22, no. 4, pp. 58–67 (in Russian). https://doi.org/10.46698/s8185-4696-7282-p
19. Lindblom O., Euler N. Solutions of discrete-velocity Boltzmann equations via Bateman and Riccati equations. *Theoretical and Mathematical Physics*, 2002, vol. 131, no. 2, pp. 595–608. https://doi.org/10.1023/A:1015428229008
20. Dukhnovskii S. A. Asymptotic stability of equilibrium states for Carleman and Godunov–Sultangazin systems of equations. *Moscow University Mathematics Bulletin*, 2019, vol. 74, no. 6, pp. 246–248. https://doi.org/10.3103/S0027132219060068
21. Vasil'eva O. A., Dukhnovskii S. A., Radkevich E. V. On the nature of local equilibrium in the Carleman and Godunov–Sultangazin equations. *Journal of Mathematical Sciences*, 2018, vol. 235, pp. 392–454. https://doi.org/10.1007/s10958-018-4080-x

Поступила в редакцию / Received 16.02.2021
Принята к публикации / Accepted 28.10.2021
Опубликована / Published 31.03.2022