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Article

Generalized model of nonlinear elastic foundation and longitudinal waves in cylindrical shells

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Abstract. A non-integrable quasi-hyperbolic sixth-order equation is derived that simulates the axisymmetric propagation of longitudinal waves along the generatrix of a cylindrical Kirchhoff – Love shell interacting with a nonlinear elastic medium. A six-parameter generalized model of a nonlinear elastic medium, which is reduced in particular cases to the models of Winkler, Pasternak, and Hetenyi, is introduced into consideration. The equation was derived by the asymptotic multiscale expansions method under the assumption that the dimensionless parameters of nonlinearity, dispersion, and thinness have the same order of smallness. The use of the introduced model made it possible to reveal additional high-frequency and low-frequency dispersions characterizing the response of the external environment to bending and shear. It is shown that non-classical shell theories should be used to reveal nonlinear effects that compensate for dispersion. It was found that the Pasternak model admits a “dispersionless” state when the dispersion due to the inertia of normal displacement is compensated by the dispersion generated by the reaction of the nonlinear elastic foundation to shear.

Keywords: cylindrical shell, axisymmetric waves, nonlinear elastic foundation, high-frequency dispersion, asymptotic integration

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Обобщенная модель нелинейно-упругого основания и продольные волны в цилиндрических оболочках

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Аннотация. Выведено неинтегрируемое квазигиперболическое уравнение шестого порядка, моделирующее осесимметричное распространение продольных волн вдоль образующей цилиндрической оболочки Кирхгофа – Лява, взаимодействующей с нелинейно-упругой средой. Введена в рассмотрение шестипараметрическая обобщенная модель нелинейно-упругой среды, сводящаяся в частных случаях к моделям Винклера, Пастернака и Хетеньи. Вывод уравнения осуществлен асимптотическим методом многих масштабов в предположении, что безразмерные параметры нелинейности, дисперсии и тонкостенности имеют одинаковый порядок малости. Использование введенной модели позволило выявить дополнительные высокочастотные и низкочастотную дисперсии, характеризующие реакцию внешней среды на изгиб и сдвиг. Показано, что для выявления нелинейных эффектов, компенсирующих дисперсию, необходимо использовать неклассические теории оболочек. Установлено, что модель Пастернака допускает «бездисперсионное» состояние, когда дисперсия, обусловленная инерцией нормального перемещения, компенсируется дисперсией, порождаемой реакцией нелинейно-упругого основания на сдвиг.

Ключевые слова: цилиндрическая оболочка, осесимметричные волны, нелинейно-упругое основание, высокочастотная дисперсия, асимптотическое интегрирование

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Introduction

The need to study and take into account the influence of an elastic foundation on the statics and dynamics of deformable systems was realized more than a hundred years ago from an analysis of the problems put forward by construction practice. The theoretical foundations for modeling the interaction of structures with an elastic foundation are contained in [1–5]. In the monograph [6], published in 1960 and has already become a classic, a wide class of problems on the refinement of the calculation schemes of the foundation and the development of simplified methods for calculating structures on an elastic foundation were solved. In recent decades, numerous results have been obtained, indicating that the interaction with the surrounding elastic (nonlinear-elastic) medium must be taken into account when studying the static and dynamic stability of structures [7–9]. The procedure for the formal construction of elastic foundation models is described in [10]. An extensive bibliography is given in the review [11] of theoretical models of elastic and viscoelastic foundations used in the analysis of oscillatory systems. The review [12] discusses the importance of using the Winkler base model in problems of adhesive mechanics and "soft matter". When solving problems of nonlinear wave dynamics of deformable systems, taking into account the influence of an external elastic medium leads to a complication of mathematical models, but allows one to identify new effects that are used in acoustic diagnostics and non-destructive testing of materials. In [13, 14], in a linear formulation, edge bending waves in a Kirchhoff plate interacting with the elastic foundations of Winkler and Pasternak were investigated. In [15], the features of localization in a Bernoulli – Euler beam on an inhomogeneous elastic foundation are analyzed. It is shown that the existence of a localized solution to the dynamic problem caused by the weakening of the stiffness of the foundation leads to a local loss of stability in statics. The phenomenon of localization of nonlinear waves in elastic bodies with inclusions is studied in [16]. It has been established that the cubic nonlinearity of the elastic foundation does not eliminate the localization phenomena and does not distort the shapes of localized waves, but leads to the dependence of the frequency of oscillations of the localized wave on the amplitude. In [17], the dispersion and spatial localization of flexural waves in a Timoshenko beam lying on a nonlinear elastic foundation were investigated. Localized longitudinal and flexural waves in a rod interacting with a nonlinear elastic medium are considered in [18]. In [19], as a result of an analysis of an axisymmetric wave process in a cylindrical Kirchhoff – Love shell, it was shown that for the existence of exact solitary wave solutions and the development of modulation instability, a soft type of nonlinearity of the external elastic medium is required. A generalization of the Winkler model, taking into account the reaction of the base in the longitudinal direction, was introduced in [20]. In the same place, for longitudinal waves in a reinforced cylindrical shell, the generalized Shamel – Ostrovsky equation is derived, and solitary wave, periodic and compact solutions are constructed [21].

This article, devoted to the derivation and analysis of an equation that simulates the axisymmetric propagation of longitudinal waves in a cylindrical shell interacting with an external nonlinear elastic medium, is organized as follows. In the first section, a nonlinear quasi-hyperbolic equation is derived from the equations of motion of an element of the Kirchhoff – Love shell using the asymptotic method of multiscale expansions. A multiparameter model of a nonlinear elastic medium is introduced, which generalizes the classical models of Winkler, Pasternak, and Hetenyi. In the second section, we briefly analyze important special cases of the derived equation. In conclusion, the results obtained are discussed.

1. Derivation of a nonlinear quasi-hyperbolic equation

The initial object of research is the equations of motion of an element of an infinite cylindrical Kirchhoff – Love shell [22], interacting with a nonlinear elastic medium. An axisymmetric case is considered, which ignores the dependence on the circumferential coordinate:

$$\frac{\gamma h}{g}\frac{\partial^2 u}{\partial t^2}-\frac{\partial N_x}{\partial x}=-f_{u1}u, \tag{1}$$

$$\frac{\gamma h}{g}\frac{\partial^2 w}{\partial t^2}-\frac{\partial^2 M_x}{\partial x^2}-\frac{1}{R}N_y-\frac{\partial}{\partial x}\left(N_x\frac{\partial w}{\partial x}\right)=$$
$$=-\left(f_{w1}w+f_{w2}w^2+f_{w3}w^3-g_p\frac{\partial^2 w}{\partial x^2}+g_h\frac{\partial^4 w}{\partial x^4}\right). \tag{2}$$

The coordinate axes Ox, Oy, and Oz are directed respectively along the longitudinal axis of the shell, along the circumference of its cross-section, and along its radius towards the cross-section center. The motions of the shell's middle surface along the axes Ox and Oz are designated as u and w. The other designations shall be as follows: N_x and M_x are normal force and bending moment in the cross-section of the shell, N_y is normal force in the longitudinal section of the shell, γ is the specific gravity of the shell material, R and h are the radii of curvature of the shell and thickness of its wall, g is the gravitational acceleration; t is time; f_{w1}, f_{w2}, f_{w3} are coefficients characterizing the resistance of the external medium during its normal deformation, f_{u1} is coefficient characterizing the resistance of the external medium during its shear deformation, g_p, g_h are the coefficients of the Pasternak and Hetenyi elastic foundation models, taking into account, respectively, the shear and bending stiffnesses of the external medium layer. Thus, a generalized six-parameter model of an external nonlinearly elastic medium is introduced into consideration, which is reduced in particular cases to the models of Winkler, Pasternak, and Hetenyi. On the right-hand sides of the system (1)–(2), there are terms responsible for the influence of the external environment on the shell dynamics and are absent in the case of the classical model.

Taking into account physical relations

$$N_x=\frac{hE}{1-\mu^2}\left(\varepsilon_x+\mu\varepsilon_y\right),\quad N_y=\frac{hE}{1-\mu^2}\left(\varepsilon_y+\mu\varepsilon_x\right),\quad M_x=\frac{h^3E}{12\left(1-\mu^2\right)}k_x, \tag{3}$$

where $\varepsilon_x,\varepsilon_y$ are deformations along axes Ox and Oy, k_x is the parameter of curvature change, E and μ are Young's modulus and Poisson's ratio of the shell material, together with the equations for the connection of deformations with displacements $\varepsilon_x=\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2$, $\varepsilon_y=-\frac{w}{R}$, $k_x=-\frac{\partial^2 w}{\partial x^2}$, we obtain the equations of motion of the shell element in displacements:

$$\frac{\gamma h}{g}\frac{\partial^2 u}{\partial t^2}-\frac{Eh}{1-\mu^2}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial w}{\partial x}\left(\frac{\partial^2 w}{\partial x^2}-\frac{\mu}{R}\right)\right)=-f_{u1}u, \tag{4}$$

$$\frac{\gamma h}{g}\frac{\partial^2 w}{\partial t^2}+\left(g_h+\frac{h^3E}{12\left(1-\mu^2\right)}\right)\frac{\partial^4 w}{\partial x^4}-g_p\frac{\partial^2 w}{\partial x^2}-\frac{hE\mu}{R\left(1-\mu^2\right)}\left(\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2+\frac{\partial u}{\partial x}-\frac{1}{\mu R}w\right)-$$
$$-\frac{hE}{1-\mu^2}\frac{\partial}{\partial x}\left[\left(\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2+\frac{\partial u}{\partial x}-\frac{\mu}{R}w\right)\frac{\partial w}{\partial x}\right]=-\left(f_{w1}w+f_{w2}w^2+f_{w3}w^3\right). \tag{5}$$

After the transition to dimensionless variables

$$U=\frac{u}{A},\quad W=\frac{w}{h},\quad X=\frac{x}{l},\quad T=\sqrt{\frac{Eg}{\gamma\left(1-\mu^2\right)}}\frac{t}{l}, \tag{6}$$

where scaling factors A and l play the role of the amplitude of longitudinal displacement and the characteristic wavelength of the disturbance, respectively, in the system of equations (4), (5) dimensionless combinations of parameters $\frac{A}{l}$, $\sqrt{\frac{hR}{l^2}}$ and $\frac{h}{R}$ are revealed, which characterize the nonlinearity of the wave process, its dispersion and the thinness of the shell. We will consider the case when the parameters of nonlinearity, dispersion, and thinness are of the same order of smallness:

$$\frac{A}{l} = \sqrt{\frac{hR}{l^2}} = \frac{h}{R} = \varepsilon \ll 1. \tag{7}$$

Thus, we consider a thin-walled shell ($h \ll R$), in which long ($R \ll l$) longitudinal ($h \ll A$) waves of small amplitude ($A \ll R$) propagate. Introducing dimensionless parameters

$$\begin{gathered} F_{u1} = \frac{R}{E\mu^2\varepsilon^3} f_{u1}, \quad G_h = \frac{1}{ER^3} g_h, \quad G_p = \frac{1}{ER\varepsilon} g_p, \\ F_{w1} = \frac{R}{E\varepsilon^2} f_{w1}, \quad F_{w2} = \frac{2\mu R^2}{E\varepsilon} f_{w2}, \quad F_{w3} = \frac{3\mu^2 R^3}{E} f_{w3}, \end{gathered} \tag{8}$$

we pass to the slow-time coordinate system $\xi = X - C_0 T$, $\tau = \varepsilon T$. Finally, in accordance with the multiscale method, we represent the dependent variables as the sum of the main part and the small correction

$$U(\xi, \tau) = U_0(\xi, \tau) + \varepsilon U_1(\xi, \tau), \quad W(\xi, \tau) = W_0(\xi, \tau) + \varepsilon W_1(\xi, \tau). \tag{9}$$

Assuming that the new dependent variables (9), their derivatives with respect to ξ and τ, as well as dimensionless parameters (8), are of the order of unity, we group the terms in the equations in powers of the small parameter ε. Equating to zero the terms in the leading order, we obtain the system of linear equations

$$\left(1 - C_0^2\right) \frac{\partial^2 U_0}{\partial \xi^2} - \mu \frac{\partial W_0}{\partial \xi} = 0, \tag{10}$$

$$-\mu \frac{\partial U_0}{\partial \xi} + W_0 = 0, \tag{11}$$

the compatibility of which requires the fulfillment of equality $W_0 = \mu \frac{\partial U_0}{\partial \xi}$, which establishes the relationship between longitudinal and transverse displacements in the linear approximation, and also condition $C_0 = \sqrt{1 - \mu^2}$, from which it follows that the perturbation propagates along the shell with a rod velocity [23].

In the next order in ε, we have the system of equations

$$\frac{\partial W_1}{\partial \xi} - \mu \frac{\partial^2 U_1}{\partial \xi^2} = \frac{2\sqrt{1-\mu^2}}{\mu} \frac{\partial^2 U_0}{\partial \xi \partial \tau} - \mu\left(1 - \mu^2\right) F_{u1} U_0, \tag{12}$$

$$\begin{gathered} W_1 - \mu \frac{\partial U_1}{\partial \xi} = -\mu\left(1 - \mu^2\right) \times \\ \times \left(G_h \frac{\partial^5 U_0}{\partial \xi^5} - (G_p - 1) \frac{\partial^3 U_0}{\partial \xi^3} + F_{w1} \frac{\partial U_0}{\partial \xi} + \frac{1}{2} F_{w2} \left(\frac{\partial U_0}{\partial \xi}\right)^2 + \frac{1}{3} F_{w3} \left(\frac{\partial U_0}{\partial \xi}\right)^3 \right). \end{gathered} \tag{13}$$

Eliminating variables W_1 and U_1 from system (12), (13), we obtain

$$\frac{2}{\mu^2\sqrt{1-\mu^2}} \frac{\partial^2 U_0}{\partial \xi \partial \tau} + G_h \frac{\partial^6 U_0}{\partial \xi^6} - (G_p - 1) \frac{\partial^4 U_0}{\partial \xi^4} +$$

$$+\left(F_{w1}+F_{w2}\frac{\partial U_0}{\partial \xi}+F_{w3}\left(\frac{\partial U_0}{\partial \xi}\right)^2\right)\frac{\partial^2 U_0}{\partial \xi^2}-F_{u1}U_0=0. \tag{14}$$

The derived equation (14) is a sixth-order nonlinear quasi-hyperbolic equation, containing, in addition to the traditional gradient-type terms, the term with the component of the desired displacement field, which characterizes the resistance of the external medium during its shear deformation. For the first time, such an additional term appeared in the already classic article by L. A. Ostrovsky [24] devoted to the study of nonlinear waves in a rotating ocean. The presence of this term, which introduces additional low-frequency dispersion into the model, significantly complicates the analytical structure of the equation and does not allow obtaining exact solitary wave and periodic solutions. The derived non-integrable equation (14) can be called the sixth order generalized Boussinesq – Ostrovsky equation.

2. Special cases of equation (14)

1. The classical model of Winkler's linear elastic medium. In this case $F_{u1}=F_{w2}=$ $=F_{w3}=G_h=G_p=0$, and equation (14) becomes linear. This means that when studying longitudinal waves in shells, the geometric nonlinearity of the classical Kirchhoff – Love model does not affect the wave process. This conclusion is valid on a time interval of the order of $1/\varepsilon$.

2. Nonlinear-elastic Winkler – Pasternak – Hetenyi foundation ($F_{u1}=0$). Equation (14) in this case contains three dispersion terms: to the traditional dispersion caused by the inertia of normal displacement, two additional ones are added, due to the reaction of the elastic foundation to shear and bending. With respect to the component of the longitudinal deformation $\frac{\partial U_0}{\partial \xi}$ equation (14) takes the form of the modified Kawahara equation ((14) without the last term), which has classes of exact periodic and solitary-wave solutions [25]. Here the problem of their physical realizability becomes the most important. In [26] it is rightly noted that "... many equations have solutions that are unsuitable from the point of view of common sense. For example, taking into account the higher-order dispersion leads to the fact that the group velocity of small-scale perturbations becomes greater than the phase velocity, while in the original equations the group velocity is always less than the phase velocity. In addition, such equations often have additional solutions that do not correspond to the known data on waves in the framework of equations complete in nonlinearity."

3. The bending stiffness of the external medium layer is not taken into account ($G_h=0$). The higher-order dispersion disappears in equation (14). The result is a combination of the classical and modified Ostrovsky equations — the so-called Gardner – Ostrovsky equation

$$\frac{\partial}{\partial \xi}\left(\frac{2}{\mu^2\sqrt{1-\mu^2}}\frac{\partial U_0}{\partial \tau}+(1-G_p)\frac{\partial^3 U_0}{\partial \xi^3}+F_{w1}\frac{\partial U_0}{\partial \xi}+\frac{F_{w2}}{2}\left(\frac{\partial U_0}{\partial \xi}\right)^2+\frac{F_{w3}}{3}\left(\frac{\partial U_0}{\partial \xi}\right)^3\right)= \\ =F_{u1}U_0. \tag{15}$$

Recently, there has been a significant increase in interest in the study of its exact and approximate solutions [27]. In this case, the external medium can be called the nonlinear elastic Pasternak medium, which, in contrast to the Winkler medium, allows one to reveal a new property of equation (14). The high-frequency dispersion coefficient can be positive, negative, or even zero, depending on the value of the coefficient G_p. In the "dispersionless" case, the reduced Ostrovsky equation

$$\frac{\partial}{\partial \xi}\left(\frac{2}{\mu^2\sqrt{1-\mu^2}}\frac{\partial U_0}{\partial \tau}+F_{w1}\frac{\partial U_0}{\partial \xi}+\frac{F_{w2}}{2}\left(\frac{\partial U_0}{\partial \xi}\right)^2+\frac{F_{w3}}{3}\left(\frac{\partial U_0}{\partial \xi}\right)^3\right)=F_{u1}U_0 \qquad (16)$$

is obtained, which has integrable reductions and exact localized solutions [28, 29]. The possibility of "controlling" the sign of the high-frequency dispersion makes it possible to use the "antisoliton theorem" [30] and to identify regimes that allow stable propagation of soliton-like perturbations.

3. Discussion and conclusions

When studying the propagation of axisymmetric longitudinal waves in elastic thin shells on the basis of the geometrically nonlinear Kirchhoff – Love model, it is not possible to analytically describe deformation solitons. In other words, asymptotic integration of the equations of motion of a shell element does not allow one to obtain an equation of the Korteweg – de Vries type for the longitudinal deformation component. This is due to the absence in the equations of motion of terms with quadratic (so-called hydrodynamic) nonlinearity such as U_x^2 and U_xU_{xx}. It is obvious that the use of geometrically nonlinear equations of the classical model of the Timoshenko type for the analysis of longitudinal waves leads to similar results. Consequently, an equation containing the necessary nonlinear terms can only be obtained using nonclassical shell theories based on refined models. In this article, nonlinearity, which compensates for dispersion and contributes to the formation of soliton-like longitudinal waves, is introduced on the basis of a generalized model of a nonlinear elastic foundation. Taking into account the fact that in shells, in contrast to plates and rods, longitudinal and normal displacements are connected already in the linear approximation, both equations of motion (1)–(2) contain terms that characterize the resistance of the external medium. The introduced model in particular cases is reduced to the linear and nonlinear models of Winkler, Pasternak, and Hetenyi. The use of the Pasternak model makes it possible to control the sign in front of high-frequency dispersion and to reveal the conditions for the propagation of solitary waves. In this case, a dispersionless state becomes possible, when the dispersion due to the inertia of normal displacement is compensated by the dispersion generated by the reaction of the nonlinear elastic foundation to shear. A brief analysis of special cases of the generalized Boussinesq – Ostrovsky equation (14) showed that the choice of a nonlinear elastic foundation model has a significant effect on the resulting wave pattern, and the problem of physical realizability of the corresponding exact solutions comes to the fore.

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