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Article

# Forcing total outer connected monophonic number of a graph

**K. Ganesamoorthy**[1✉], **S. Lakshmi Priya**[2]

[1]Coimbatore Institute of Technology, Department of Mathematics, Coimbatore — 641 014, India
[2]CIT Sandwich Polytechnic College, Department of Mathematics, Coimbatore — 641 014, India
**Kathiresan Ganesamoorthy**, kvgm_2005@yahoo.co.in, https://orcid.org/0000-0003-2769-1991
**Shanmugam Lakshmi Priya**, lakshmiuspriya@gmail.com, https://orcid.org/0000-0001-7367-1532

**Abstract.** For a connected graph $G = (V, E)$ of order at least two, a subset $T$ of a minimum total outer connected monophonic set $S$ of $G$ is a *forcing total outer connected monophonic subset* for $S$ if $S$ is the unique minimum total outer connected monophonic set containing $T$. A forcing total outer connected monophonic subset for $S$ of minimum cardinality is a *minimum forcing total outer connected monophonic subset* of $S$. The *forcing total outer connected monophonic number* $f_{tom}(S)$ in $G$ is the cardinality of a minimum forcing total outer connected monophonic subset of $S$. The *forcing total outer connected monophonic number* of $G$ is $f_{tom}(G) = \min\{f_{tom}(S)\}$, where the minimum is taken over all minimum total outer connected monophonic sets $S$ in $G$. We determine bounds for it and find the forcing total outer connected monophonic number of a certain class of graphs. It is shown that for every pair $a, b$ of positive integers with $0 \leqslant a < b$ and $b \geqslant a + 4$, there exists a connected graph $G$ such that $f_{tom}(G) = a$ and $cm_{to}(G) = b$, where $cm_{to}(G)$ is the total outer connected monophonic number of a graph.

**Keywords:** total outer connected monophonic set, total outer connected monophonic number, forcing total outer connected monophonic subset, forcing total outer connected monophonic number

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# Форсирование общего внешне связного монофонического числа графа

**К. Ганесамурти**[1✉], **Ш. Лакшми Прия**[2]

[1]Технологический институт Коимбатура, Индия, Коимбатур — 641 014
[2]Политехнический колледж Технологического института Коимбатура, Индия, Коимбатур — 641 014
**Катиресан Ганесамурти**, Dr., доцент математического факультета, kvgm_2005@yahoo.co.in, https://orcid.org/0000-0003-2769-1991
**Шанмугам Лакшми Прия**, студентка математического факультета, lakshmiuspriya@gmail.com, https://orcid.org/0000-0001-7367-1532

**Аннотация.** Для связного графа $G = (V, E)$ с числом вершин не менее 2 подмножество $T$ минимального общего внешне связного монофонического множества $S$ графа $G$ является *сильным общим внешне связным монофоническим подмножеством* для $S$, если $S$ есть единственное минимальное общее внешне связное монофоническое множество, содержащее $T$. Сильное общее внешне связное монофоническое подмножество для $S$ с минимальным числом элементов есть *минимальное сильное общее внешне связное монофоническое подмножество* $S$. *Сильное общее внешне связное монофоническое число* $f_{tom}(S)$ в $G$ есть число элементов минимального сильного общего внешне связного монофонического подмножества $S$. *Сильное общее внешне связное монофоническое число* графа $G$ есть $f_{tom}(G) = \min\{f_{tom}(S)\}$, где минимум принимается над всеми минимальными общими внешне связными монофоническими множествами $S$ в $G$. Мы определяем его границы и находим сильное общее внешне связное монофоническое число некоторых классов графов. Показывается, что для каждой пары $a$, $b$ положительных целых с $0 \leqslant a < b$ и $b \geqslant a + 4$ существует связный граф $G$ такой, что $f_{tom}(G) = a$ и $cm_{to}(G) = b$, где $cm_{to}(G)$ является общим внешне связным монофоническим числом графа.
**Ключевые слова:** общее внешне связное монофоническое множество, общее внешне связное монофоническое число, сильное общее внешне связное монофоническое подмножество, сильное общее внешне связное монофоническое число
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## Introduction

By a graph $G = (V, E)$ we mean a finite simple undirected connected graph. The order and size of $G$ are denoted by $p$ and $q$, respectively. For basic graph theoretic terminology we refer to Harary [1,2]. The *distance* $d(x, y)$ between two vertices $x$ and $y$ in a connected graph $G$ is the length of a shortest $x - y$ path in $G$. An $x - y$ path of length $d(x, y)$ is called an $x - y$ *geodesic*. A vertex $v$ of a connected graph $G$ is called an *endvertex* of $G$ if its degree is 1. A vertex $v$ of a connected graph $G$ is called a *support vertex* of $G$ if it is adjacent to an endvertex of $G$. The *neighborhood* of a vertex $v$ is the set $N(v)$ consisting of all vertices $u$ which are adjacent with $v$. A vertex $v$ is an *extreme vertex* if the subgraph induced by its neighbors is complete. A *chord* of a path $P$ is an edge joining two non-adjacent vertices of $P$. A path $P$ is called a *monophonic path* if it is a chordless path. A set $S$ of vertices of $G$ is a *monophonic set* of $G$ if each vertex $v$ of $G$ lies on a $x - y$ monophonic path for some $x$ and $y$ in $S$. The minimum cardinality of a monophonic set of $G$ is the *monophonic number* of $G$ and is denoted by $m(G)$. The monophonic number of a graph, an algorithmic aspect of monophonic concepts was introduced and studied in [3–7]. A *total monophonic set* of a graph $G$ is a monophonic set $S$ such that the subgraph $G[S]$ induced by $S$ has no isolated vertices. The minimum cardinality of a total monophonic set of $G$ is the *total monophonic number* of $G$ and is denoted by $m_t(G)$. The total monophonic number of a graph and its related concepts were studied in [8–10]. A set $S$ of vertices in a graph $G$ is said to be an *outer connected monophonic set* if $S$ is a monophonic set of $G$ and either $S = V$ or the subgraph induced by $V - S$ is connected. The minimum cardinality of an outer connected monophonic set of $G$ is the *outer connected monophonic number* of $G$ and is denoted by $m_{oc}(G)$. The outer connected monophonic number of a graph was introduced in [11]. Very recently, outer connected monophonic concepts have been widely investigated in graph theory, such as a connected outer connected monophonic number [12], extreme outer connected monophonic graphs [13], and so on. A *total outer connected monophonic set* $S$ of $G$ is an outer connected monophonic set such that the subgraph induced by $S$ has no isolated vertices. The minimum cardinality of a total outer connected monophonic set of $G$ is the *total outer connected monophonic number* of $G$ and is denoted by $cm_{to}(G)$.

The authors of this article introduced and studied the general externally total outer connected monophonic number of a graph and proved the following theorems[1], which will be used further.

**Theorem 1.** *Each extreme vertex and each support vertex of a connected graph $G$ belong to every total outer connected monophonic set of $G$.*

**Theorem 2.** *For the complete graph $K_p (p \geqslant 2)$, $cm_{to}(K_p) = p$.*

**Theorem 3.** *For any non-trivial tree $T$, the set of all endvertices and support vertices of $T$ is the unique minimum total outer connected monophonic set of $G$.*

[1]Ganesamoorthy K., Lakshmi Priya S. The total outer connected monophonic number of a graph. *Transactions of A. Razmadze Mathematical Institute*, accepted.

**Theorem 4.** *For any connected graph $G$, $cm_{to}(G) = 2$ if and only if $G = K_2$.*

Throughout this paper, $G$ denotes a connected graph with at least two vertices.

## 1. Main Results

**Definition 1.** Let $S$ be a minimum total outer connected monophonic set of $G$. A subset $T$ of $S$ is a *forcing total outer connected monophonic subset* for $S$ if $S$ is the unique minimum total outer connected monophonic set containing $T$. A forcing total outer connected monophonic subset for $S$ of minimum cardinality is a *minimum forcing total outer connected monophonic subset* of $S$. The *forcing total outer connected monophonic number* $f_{tom}(S)$ in $G$ is the cardinality of a minimum forcing total outer connected monophonic subset of $S$. The *forcing total outer connected monophonic number* of $G$ is $f_{tom}(G) = \min\{f_{tom}(S)\}$, where the minimum is taken over all minimum total outer connected monophonic sets $S$ in $G$.

**Example 1.** For the graph $G$ in Fig. 1, it is clear that $S_1 = \{v_1, v_2, v_4, v_5\}$, $S_2 = \{v_1, v_4, v_5, v_8\}$, $S_3 = \{v_1, v_2, v_5, v_6\}$ and $S_4 = \{v_1, v_5, v_6, v_8\}$ are the minimum total outer connected monophonic sets of $G$. It is clear that no minimum total outer connected monophonic set $S_i (i = 1, 2, 3, 4)$ is the unique minimum total outer connected monophonic set containing any of its 1-element subsets. It is easy to see that $\{v_2, v_4\}$ is a forcing total outer connected monophonic subset contained in $S_1$ and $f_{tom}(S_1) = 2$. Hence, we have $f_{tom}(G) = 2$. By Theorem 3, for any non-trivial tree $T$, the set of all endvertices and support vertices of $T$ is the unique minimum total outer connected monophonic set of $T$ and so $f_{tom}(T) = 0$.

**Theorem 5.** *For any connected graph $G$ of order $p$, $0 \leqslant f_{tom}(G) \leqslant cm_{to}(G) \leqslant p$.*

**Proof.** By the definition of the forcing total outer connected monophonic number of a graph, it is clear that $f_{tom}(G) \geqslant 0$. Let $S$ be a minimum total outer connected monophonic set of $G$. Clearly, $f_{tom}(S) \leqslant |S| = cm_{to}(G)$ and $f_{tom}(G) = \min\{f_{tom}(S)\}$, where the minimum is taken over all minimum total outer connected monophonic sets $S$ in $G$. Hence $0 \leqslant f_{tom}(G) \leqslant cm_{to}(G) \leqslant p$. □

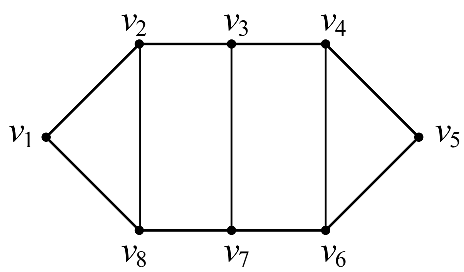

Fig. 1. Graph $G$ with $f_{tom}(G) = 2$

**Remark 1.** The bounds in Theorem 5 are sharp. By Theorem 3, for any non-trivial tree $T$, the set of all endvertices and support vertices of $T$ is the unique minimum total outer connected monophonic set of $T$ and so $f_{tom}(T) = 0$. By Theorem 2, for the complete graph $K_p (p \geqslant 2)$, $cm_{to}(K_p) = p$. Also all the inequalities in Theorem 5 can be strict. For the graph $G$ given in Fig. 1 of order 8, it is clear that no 2-element subset or 3-element subset of $V(G)$ is a total outer connected monophonic set of $G$. The minimum total outer connected monophonic sets of $G$ are $S_1 = \{v_1, v_2, v_4, v_5\}$, $S_2 = \{v_1, v_4, v_5, v_8\}$, $S_3 = \{v_1, v_2, v_5, v_6\}$ and $S_4 = \{v_1, v_5, v_6, v_8\}$ so that $cm_{to}(G) = 4$. It is clear that $f_{tom}(S_i) = 2 (i = 1, 2, 3, 4)$ and so $f_{tom}(G) = 2$. Thus $0 < f_{tom}(G) < cm_{to}(G) < p$.

The following theorem characterizes graphs $G$ for which the lower bound in Theorem 5 is attained and also characterizes graphs $G$ for which $f_{tom}(G) = 1$ and $f_{tom}(G) = cm_{to}(G)$.

**Theorem 6.** *Let $G$ be a connected graph. Then*

*(i) $f_{tom}(G) = 0$ if and only if $G$ has the unique minimum total outer connected monophonic set;*

*(ii) $f_{tom}(G) = 1$ if and only if $G$ has at least two minimum total outer connected monophonic sets, one of which is the unique minimum total outer connected monophonic set containing one of its elements;*

*(iii) $f_{tom}(G) = cm_{to}(G)$ if and only if no minimum total outer connected monophonic set of $G$ is the unique minimum total outer connected monophonic set containing any of its proper subsets.*

**Proof.** (i) Let $f_{tom}(G) = 0$. Then, by the definition, $f_{tom}(S) = 0$ for some minimum total outer connected monophonic set $S$ of $G$ so that the empty set $\varphi$ is the minimum forcing subset for $S$. Since the empty set $\varphi$ is a subset of every set, it follows that $S$ is the unique minimum total outer connected monophonic set of $G$. The converse is clear.

(ii) Let $f_{tom}(G) = 1$. Then by (i), $G$ has at least two minimum total outer connected monophonic sets. Since $f_{tom}(G) = 1$, there is a 1-element subset $T$ of a minimum total outer connected monophonic set $S$ of $G$ such that $T$ is not a subset of any other minimum total outer connected monophonic set of $G$. Thus $S$ is the unique minimum total outer connected monophonic set containing one of its elements. The converse is clear.

(iii) Let $f_{tom}(G) = cm_{to}(G)$. Then $f_{tom}(S) = cm_{to}(G)$ for every minimum total outer connected monophonic set $S$ in $G$. Since any total outer connected monophonic set of $G$ needs at least two vertices, $cm_{to}(G) \geqslant 2$ and hence $f_{tom}(G) \geqslant 2$. Then by (i), $G$ has at least two minimum total outer connected monophonic sets, and so the empty set $\varphi$ is not a forcing subset for any minimum total outer connected monophonic set of $G$. Since $f_{tom}(G) = cm_{to}(G)$, no proper subset of $S$ is a forcing subset of $S$. Thus no minimum total outer connected monophonic set of $G$ is the unique minimum total outer connected monophonic set containing any of its proper subsets.

Conversely, the data implies that $G$ contains more than one minimum total outer connected monophonic set, and no subset of any minimum total outer connected monophonic set $S$ other than $S$, is a forcing subset for $S$. Hence it follows that $f_{tom}(G) = cm_{to}(G)$. □

**Definition 2.** A vertex $v$ of $G$ is said to be a *total outer connected monophonic vertex* if $v$ belongs to every minimum total outer connected monophonic set of $G$.

**Remark 2.** If $G$ has the unique minimum total outer connected monophonic set $S$, then every vertex in $S$ is a total outer connected monophonic vertex of $G$. Also, if $x$ is an extreme vertex or a support vertex of $G$, then $x$ is a total outer connected monophonic vertex of $G$. For the graph $G$ given in Fig. 1, $v_1$ and $v_5$ are the total outer connected monophonic vertices of $G$.

The next theorem and corollary are an immediate consequence of the definitions of total outer connected monophonic vertex and a forcing total outer connected monophonic subset of $G$.

**Theorem 7.** *Let $G$ be a connected graph and let $\Psi_{tom}$ be the set of relative complements of the minimum forcing total outer connected monophonic subsets in their respective minimum total outer connected monophonic sets in $G$. Then $\cap_{F\in\Psi_{tom}} F$ is the set of all total outer connected monophonic vertices of $G$.*

**Corollary 1.** *Let $S$ be a minimum total outer connected monophonic set of $G$. Then no total outer connected monophonic vertex of $G$ belongs to any minimum forcing total outer connected monophonic subset of $S$.*

**Theorem 8.** *Let $M$ be the set of all total outer connected monophonic vertices of $G$. Then $f_{tom}(G) \leqslant cm_{to}(G) - |M|$.*

**Proof.** Let $S$ be any minimum total outer connected monophonic set of $G$. Then $cm_{to}(G) = |S|$, $M \subseteq S$, and $S$ is the unique minimum total outer connected monophonic set containing $S - M$. Hence $f_{tom}(G) \leqslant |S - M| = |S| - |M| = cm_{to}(G) - |M|$. ☐

**Corollary 2.** *If $G$ is a connected graph with $l$ extreme vertices and $k$ support vertices, then $f_{tom}(G) \leqslant cm_{to}(G) - (l + k)$.*

**Remark. 3.** The bound in Theorem 8 is sharp. For the graph $G$ given in Fig. 1, the minimum total outer connected monophonic sets of $G$ are $S_1 = \{v_1, v_2, v_4, v_5\}$, $S_2 = \{v_1, v_4, v_5, v_8\}$, $S_3 = \{v_1, v_2, v_5, v_6\}$ and $S_4 = \{v_1, v_5, v_6, v_8\}$ so that $cm_{to}(G) = 4$. It is clear that $f_{tom}(S_i) = 2(i = 1, 2, 3, 4)$ and so $f_{tom}(G) = 2$. Also, $M = \{v_1, v_5\}$ is the set of all total outer connected monophonic vertices of $G$ and so $f_{tom}(G) = cm_{to}(G) - |M|$. The inequality in Theorem 8 can be strict. For the graph $G$ given in Fig. 2, the minimum total outer connected monophonic sets of $G$ are $M_1 = \{v_1, v_2, v_3, v_6\}$, $M_2 = \{v_3, v_4, v_5, v_6\}$, $M_3 = \{v_2, v_3, v_4, v_6\}$ and so $cm_{to}(G) = 4$. It is clear that $f_{tom}(M_i) = 1 \quad (i = 1, 2)$, and so $f_{tom}(G) = 1$. Also, the vertices $v_3$ and $v_6$ are the total outer connected monophonic vertices of $G$, we have $f_{tom}(G) < cm_{to}(G) - |M|$.

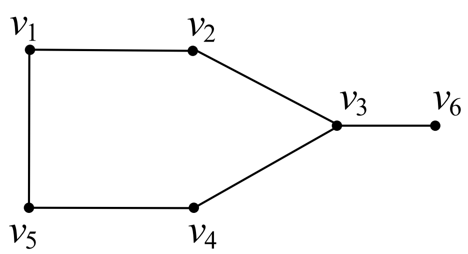


Fig. 2. A graph $G$ with $f_{tom}(G) <$ $< cm_{to}(G) - |M|$

**Theorem 9.** *If $G$ is a connected graph with $cm_{to}(G) = 2$, then $f_{tom}(G) = 0$.*

**Proof.** If $cm_{to}(G) = 2$ then by Theorem 4, we have $G = K_2$. Hence $V(G)$ is the unique minimum total outer connected monophonic set of $G$. Also, by Theorem 6($i$), $f_{tom}(G) = 0$. ☐

**Remark 4.** The converse of Theorem 9 need not be true. For the path $P_4$ of order 4, the vertex set $V(P_4)$ is the unique minimum total outer connected monophonic set of $G$ and so $cm_{to}(P_4) = 4$. By Theorem 6 (i), $f_{tom}(P_4) = 0$.

**Theorem 10.** *For the complete bipartite graph $G = K_{m,n}(2 \leqslant m \leqslant n)$,*

$$f_{tom}(G) = \begin{cases} m + n - 1 & \text{if } 2 = m \leqslant n, \\ 4 & \text{if } 3 \leqslant m \leqslant n. \end{cases}$$

**Proof.** Let $U = \{u_1, u_2, \ldots, u_m\}$ and $W = \{w_1, w_2, \ldots, w_n\}$ be the partite sets of $G$, where $m \leqslant n$. We prove this theorem by considering two cases.

*Case 1*. If $m = 2$, then it is clear that any minimum total outer connected monophonic sets of $G$ is of the form $V(G) - \{w_i\}(1 \leqslant i \leqslant n)$ or $V(G) - \{u_j\}(1 \leqslant j \leqslant m)$. It is easy to verify that, no minimum total outer connected monophonic set of $G$ is the unique minimum total outer connected monophonic set containing any of its proper subsets. Then by Theorem 6 (iii), we have $f_{tom}(G) = m + n - 1$.

*Case 2*. If $3 \leqslant m \leqslant n$, then any minimum total outer connected monophonic set of $G$ is obtained by choosing any two elements from $U$ as well as $W$, and $G$ has at least two minimum total outer connected monophonic sets. Hence $cm_{to}(G) = 4$. Clearly, no minimum total outer connected monophonic set of $G$ is the unique minimum total outer connected monophonic set containing any of its proper subsets. Then by Theorem 6 (iii), we have $f_{tom}(G) = cm_{to}(G) = 4$. □

**Theorem 11.** *For any cycle $C_n (n \geqslant 3)$, $f_{tom}(C_n) = \begin{cases} 0 & \text{if } n = 3, \\ 3 & \text{if } n = 4, \\ 2 & \text{if } n \geqslant 5. \end{cases}$*

**Proof.** Let $C_n : v_1, v_2, \ldots, v_n, v_1$ be a cycle of order $n$. We prove this theorem by considering two cases.

*Case 1:* $n = 3$. Since $C_3$ is the complete graph of order 3, $V(C_3)$ is the unique minimum total outer connected monophonic set of $C_3$. By Theorem 6 (i), $f_{tom}(C_3) = 0$.

*Case 2:* $n \geqslant 4$. It is clear that no 2-element subset of $V(C_n)$ is a total outer connected monophonic set of $C_n$. It is easy to verify that any minimum total outer connected monophonic set of $C_n$ consists of three consecutive vertices of $C_n$ so that $cm_{to}(C_n) = 3$. For $n = 4$, it is clear that no minimum total outer connected monophonic set of $C_4$ is the unique minimum total outer connected monophonic set containing any of its proper subsets. Thus by Theorem 6 (iii), we have $f_{tom}(C_4) = 3$. For $n \geqslant 5$, it is clear that the set of two non-adjacent vertices of any minimum total outer connected monophonic set $S$ of $C_n$ is a minimum forcing total outer connected monophonic subset of $S$ and so $f_{tom}(S) = 2$. Hence $f_{tom}(C_n) = 2$. □

**Theorem 12.** *For the wheel $W_n = K_1 + C_{n-1}$ $(n \geqslant 5)$, $f_{tom}(W_n) = \begin{cases} 3 & \text{if } n = 5, \\ 2 & \text{if } n \geqslant 6. \end{cases}$*

**Proof.** It is clear that no 2-element subset of $V(W_n)$ is a total outer connected monophonic set of $W_n$. It is easy to observe that any minimum total outer connected monophonic set of $W_n$ consists of three consecutive vertices of $C_{n-1}$ so that $cm_{to}(W_n) = 3$. For $n = 5$, it is clear that no minimum total outer connected monophonic set of $W_5$ is the unique minimum total outer connected monophonic set containing any of its proper subsets. Thus by Theorem 6 (iii), we have $f_{tom}(W_5) = 3$. For $n \geqslant 6$, it is clear that the set of two non-adjacent vertices of any minimum total outer connected monophonic set $S$ of $W_n$ is a minimum forcing total outer connected monophonic subset of $S$ and so $f_{tom}(S) = 2$. Hence $f_{tom}(W_n) = 2$. □

**Theorem 13.** *For any complete graph $G = K_p (p \geqslant 2)$ or any non-trivial tree $G = T$, $f_{tom}(G) = 0$.*

**Proof.** Let $G = K_p$. By Theorem 2, the set of all vertices of $G$ is the unique minimum total outer connected monophonic set of $G$ and so by Theorem 6 (i), $f_{tom}(G) = 0$. If $G$ is a non-trivial tree, then by Theorem 3, the set of all endvertices and support vertices of $G$ is the unique minimum total outer connected monophonic set of $G$ and by Theorem 6 (i), $f_{tom}(G) = 0$. □

**Theorem 14.** *For every pair $a$, $b$ of integers such that $0 \leqslant a < b$ and $b \geqslant a + 4$, there is a connected graph $G$ with $f_{tom}(G) = a$ and $cm_{to}(G) = b$.*

**Proof.** If $a = 0$, let $G = K_b$. Then by Theorem 13, $f_{tom}(G) = 0$, and by Theorem 2, $cm_{to}(G) = b$. Now, assume that $0 < a < b$. The required graph $G$ is obtained from the star $K_{1,4}$ having the vertex set $\{z_1, z_2, z_3, z_4, z_5\}$ with $z_3$ as the cut-vertex by adding $a + b - 2$ new vertices $w_1, w_2, \ldots, w_a, v_1, v_2, \ldots, v_a, u_1, u_2, \ldots, u_{b-a-3}, x$ and joining each $w_i (1 \leqslant i \leqslant a)$ to the vertices $z_2, z_1$ and $z_4$; and joining each $v_i$ $(1 \leqslant i \leqslant a)$ to the vertices $z_2$, $z_4$ and $z_5$; and joining each $u_i$ $(1 \leqslant i \leqslant b-a-3)$ to the vertex $z_5$; and also joining the vertex $x$ to the vertex $z_1$, the vertex $z_1$ to the vertex $z_5$, and the vertex $z_2$ to the vertex $z_4$. The graph $G$ is shown in Fig. 3. Let $S = \{u_1, u_2, \ldots, u_{b-a-3}, x, z_1, z_5\}$ be the set of all endvertices and support vertices of $G$. By Theorem 1, every total outer connected monophonic set of $G$ contains $S$. It is clear that $S$ is not a total outer connected monophonic set of $G$. We observe that every minimum total outer connected monophonic set of $G$ contains exactly one vertex from the set $\{v_i, w_i\}$ for every $(1 \leqslant i \leqslant a)$. Thus $cm_{to}(G) \geqslant b$. Since $S_1 = S \cup \{w_1, w_2, \ldots, w_a\}$ is a total outer connected monophonic set of $G$, it follows that $cm_{to}(G) = b$.

Next, we show that $f_{tom}(G) = a$. Since every minimum total outer connected monophonic set of $G$ contains $S$, it follows from Theorem 8 that $f_{tom}(G) \leqslant cm_{to}(G) - |S| = b - (b - a) = a$. It is clear that every minimum total outer connected monophonic set $S'$ of $G$ is of the form $S \cup \{x_1, x_2, \ldots, x_a\}$, where $x_i \in \{v_i, w_i\}$ for every $i$ $(1 \leqslant i \leqslant a)$. Let $T$ be any proper subset of $S'$ with $|T| < a$. Then there is a vertex $x \in S' - S$ such that $x \notin T$. If $x = v_i$ $(1 \leqslant i \leqslant a)$, then $S'' = (S' - \{v_i\}) \cup \{w_i\}$ is a minimum total outer connected monophonic set of $G$ containing $T$. Similarly, if $x = w_j (1 \leqslant j \leqslant a)$, then $S''' = (S' - \{w_j\}) \cup \{v_j\}$ is a minimum total outer connected monophonic set of $G$ containing $T$. Thus $S'$ is not the unique minimum total outer connected monophonic set containing $T$ and so $T$ is not a forcing total outer connected monophonic subset of $S'$. This is true for all minimum total outer connected monophonic sets of $G$ and so $f_{tom}(G) = a$. □

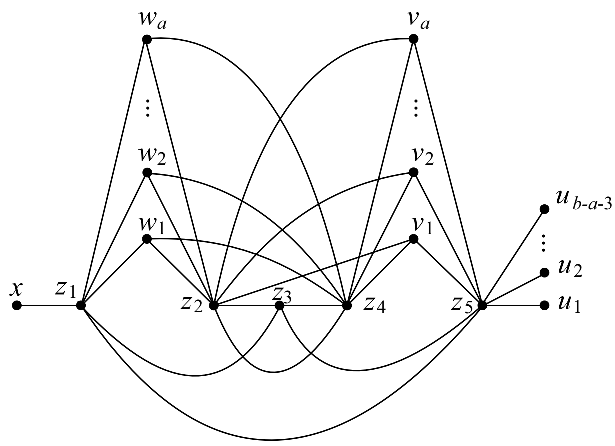


Fig. 3. A graph $G$ with $f_{tom}(G) = a > 0$ and $cm_{to}(G) = b > a$

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