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Article

New integral inequalities in the class of functions (h, m)-convex

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Abstract. In this article, we have defined new weighted integral operators. We formulated a lemma in which we obtained a generalized identity through these integral operators. Using this identity, we obtain some new generalized Simpson's type inequalities for (h, m)-convex functions. These results we obtained using the convexity property, the classical Hölder inequality, and its other form, the power mean inequality. The generality of our results lies in two fundamental points: on the one hand, the integral operator used and, on the other, the notion of convexity. The first, because the "weight" allows us to encompass many known integral operators (including the classic Riemann and Riemann – Liouville), and the second, because, under an adequate selection of the parameters, our notion of convexity contains several known notions of convexity. This allows us to show that many of the results reported in the literature are particular cases of ours.

Keywords: convex functions, (m, h)-convex functions, Simpson's type inequality, Hermite – Hadamard inequality, Hölder inequality, weighted integrals
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Новые интегральные неравенства в классе (h, m)-выпуклых функций

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Аннотация. В статье определены новые взвешенные интегральные операторы. Сформулирована лемма, в которой получено обобщенное тождество через эти интегральные операторы. С использованием данного тождества получены некоторые новые обобщенные неравенства типа Симпсона для (h, m)-выпуклых функций. Эти результаты получены на основе свойства выпуклости, классического неравенства Гельдера и его другой формы — неравенства степенного среднего. Общность результатов статьи заключается в двух основных моментах. Первый — используемый интегральный оператор, так как «вес» позволяет охватить многие известные интегральные операторы, в том числе классические Римана и Римана – Лиувилля. Второй момент — используемое понятие выпуклости, при адекватном выборе параметров оно содержит несколько уже известных понятий выпуклости. Это позволяет сделать заключение, что многие известные в литературе результаты являются частными случаями рассматриваемых в статье.

Ключевые слова: выпуклые функции, (m, h)-выпуклые функции, неравенство типа Симпсона, неравенство Эрмита – Адамара, неравенство Гельдера, взвешенные интегралы

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Introduction

The concept of convexity for a number of scientific disciplines related to mathematics (Optimization Theory, Numerical Analysis, Computational Mathematics, etc.) is the main concept since it is closely related to estimating the mean value of a function given on an interval. Today in the literature there are many different classes of convexity of a function that extend this concept. The Definition of convexity is given in the literature as follows:

Definition 1. The function $\phi : [\upsilon^*, \vartheta^*] \to \mathbb{R}$, is said to be convex if we have

$$\phi\left(\varsigma y + (1-\varsigma)x\right) \leqslant \varsigma\phi(y) + (1-\varsigma)\phi(x)$$

$\forall\ x, y \in [\upsilon^*, \vartheta^*]$ and $\varsigma \in [0, 1]$.

In [1], a fairly wide range of convexity classes and their relations are given.

In the literature, the well-known Simpson-type inequality is presented as follows.

If $\phi \in C^4(\upsilon^*, \vartheta^*)$ and $\left\|\phi^{(4)}\right\|_\infty := \sup\limits_{x\in(\upsilon^*,\vartheta^*)} \left|\phi^{(4)}(x)\right| < \infty$, then

$$\left|\frac{\vartheta^* - \upsilon^*}{3}\left[\frac{\phi(\upsilon^*) + \phi(\vartheta^*)}{2} + 2\phi\left(\frac{\upsilon^* + \vartheta^*}{2}\right)\right] - \int_{\upsilon^*}^{\vartheta^*}\phi(x)dx\right| \leqslant \frac{(\vartheta^* - \upsilon^*)^5}{2880}\left\|\phi^{(4)}\right\|_\infty. \quad (1)$$

A number of recent studies have been devoted to refinements and generalizations of Simpson's type inequalities for various classes of convex functions. For example, for the quasi-convex functions Alomari and Hussain in [2] and Set et al in [3] in terms of differentiable functions obtained some Simpson's type inequalities. Bayraktar in [4], presented Hadamard and Simpson's type parametric integral inequalities for concave and r-convex functions in terms of special means. New generalized integral inequalities of the Simpson and Hadamard type for convex functions, or functions satisfying the Lipschitz or Lagrange conditions, were obtained by the authors in [5]. In [6] Dragomir et al and Liu in [7] presented Simpson's type inequalities for the continuously differentiable functions and their application. Hussain and Qaisar in [8] established some new inequalities of Simpson's type for functions whose third derivatives are prequasiinvex and preinvex. In [9] Park presented generalized Simpson's type and Hadamard type integral inequalities for functions whose q-th powers of second derivatives are decreasing (α, m)-geometrically convex. In [10–13] authors established new inequalities of Simpson's type based on s-convexity. In [14–16] authors established new inequalities of Simpson's type for extended (s, m)-convex and generalized (s, m)-preinvex functions. Extended Simpson-type inequalities for the class of differentiable concave functions related to the Hadamard inequality were obtained by Hsu et al in [17]. Simpson's type double integral inequalities and applications for numerical integration were given by Ujević in [18].

In [19–21] we presented the following Definitions.

Definition 2. Let $h : [0, 1] \to (0, 1]$ and $\phi : X = [0, +\infty) \to [0, +\infty)$. If inequality

$$\phi\left(\varsigma\xi + m(1-\varsigma)\zeta\right) \leqslant h^s(\varsigma)\phi(\xi) + m(1 - h^s(\varsigma))\phi(\zeta) \tag{2}$$

is fulfilled for all $\xi, \zeta \in X$ and $\varsigma \in [0, 1]$, where $0 \leqslant m \leqslant 1$, $s \in (0, 1]$. Then the function ϕ will be called the (h, m)-convex modification of the first type on X.

Definition 3. Let $h : [0, 1] \to (0, 1]$ and $\phi : X = [0, +\infty) \to [0, +\infty)$. If inequality

$$\phi\left(\varsigma\xi + m(1-\varsigma)\zeta\right) \leqslant h^s(\varsigma)\phi(\xi) + m(1 - h(\varsigma))^s\phi(\zeta) \tag{3}$$

is fulfilled for all $\xi, \zeta \in X$ and $\varsigma \in [0, 1]$, where $s \in [-1, 1], 0 \leqslant m \leqslant 1$. Then the function ϕ will be called the (h, m)-convex modification of the second type on X.

Remark 1. From the Definitions above, the sets (h, m)-convex modified functions of the first and second types characterized by the triple $(h(\varsigma), m, s)$ are denoted by $N^{s,1}_{h,m}[\upsilon^*, \vartheta^*]$ and $N^{s,2}_{h,m}[\upsilon^*, \vartheta^*]$, respectively. In [20, 21] you can see the convex classes obtained from the special cases of this triple.

Remark 2. In the different notions of convexity, if the direction of the inequality changes, it will be called concave.

In our work, we use the Euler Gamma functions Γ [22] and Γ_κ [23]:

$$\Gamma(z) = \int_0^\infty \varsigma^{z-1} e^{-\varsigma}\, d\varsigma, \qquad \mathrm{Re}\,(z) > 0,$$
$$\Gamma_\kappa(z) = \int_0^\infty \varsigma^{z-1} e^{-\varsigma^\kappa/\kappa}\, d\varsigma, \qquad \kappa > 0.$$

Here $\Gamma_\kappa(z) = (\kappa)^{\frac{z}{\kappa}-1}\Gamma\left(\frac{z}{\kappa}\right)$ and $\Gamma_\kappa(z+\kappa) = z\Gamma_\kappa(z)$, and $\lim\limits_{\kappa\to 1}\Gamma_\kappa(z) = \Gamma(z)$.

To facilitate understanding of the subject of research, we first give the Definition of the Riemann – Liouville fractional integral: (with $0 \leqslant \upsilon^* < \varsigma < \vartheta^* \leqslant \infty$).

Definition 4. Let $\phi \in L_1[\upsilon^*, \vartheta^*]$. Then the Riemann – Liouville fractional integrals of order $\alpha \in \mathbb{C}$, $\mathrm{Re}\,(\alpha) > 0$ are defined by (right and left respectively):

$$^{\alpha}I_{\upsilon^*}\phi(\sigma) = \frac{1}{\Gamma(\alpha)}\int_{\upsilon^*}^{\sigma} \phi(\varsigma)(\sigma-\varsigma)^{\alpha-1}\, d\varsigma, \quad \sigma > \upsilon^* \tag{4}$$

and

$$^{\alpha}I_{\vartheta^*}\phi(\sigma) = \frac{1}{\Gamma(\alpha)}\int_{\sigma}^{\vartheta^*} \phi(\varsigma)(\varsigma-\sigma)^{\alpha-1}\, d\varsigma, \quad \sigma < \vartheta^*. \tag{5}$$

Next we present the weighted integral operators, which will be the basis of our work.

Definition 5. Let $\phi \in L([\upsilon^*, \vartheta^*])$ and let function $w \in C[0,1]$, and $w \in \mathbb{R}^+ \cup \{0\}$, with a piecewise continuous derivative on $[0,1]$. Then

$$J^w_{\upsilon^*}\phi(\sigma) = \int_{\upsilon^*}^{\sigma} \phi(\varsigma) w' \left(\frac{\sigma - \varsigma}{\sigma - \upsilon^*}\right) d\varsigma$$

and

$$J^w_{\vartheta^*}\phi(\sigma) = \int_{\sigma}^{\vartheta^*} \phi(\varsigma) w' \left(\frac{\varsigma - \sigma}{\vartheta^* - \sigma}\right) d\varsigma$$

are defined respectively as right and left weighted fractional integrals with $\upsilon^* < \sigma \leqslant \vartheta^*$.

Remark 3. Consider some particular cases of Definition 5:

1) putting $w'(\varsigma) \equiv 1$, we get the ordinary Riemann integral;

2) if $w'(\varsigma) = \frac{(\sigma-\upsilon^*)^{\alpha-1}\varsigma^{\alpha-1}}{\Gamma(\alpha)}$, then we obtain (4), and (5) can be obtained similarly;

3) with convenient kernel choices w' we can get:

– the k-Riemann – Liouville fractional integrals, from [24];

– the fractional integral (right-sided) from [25], of a function ϕ with respect to another function g on $[\upsilon^*, \vartheta^*]$;

– the right and left integral operator from [26];

– the right and left sided generalized fractional integral operators from [27];

– the integral operators from [28] and [29], can also be obtained from above Definition by imposing similar conditions to w'.

Of course, there are other known integral operators, fractional or not, that can be obtained as particular cases of the previous one, but we leave it to interested readers.

The purpose of this work is to obtain new Simpson-type inequalities through the weighted integrals of Definition 5 and to show that these results generalize a number of well-known results from the literature, including those for Hadamard-type inequalities.

1. Results

Our results are obtained using the following lemma:

Lemma 1. *Let $\phi : [\upsilon^*, \vartheta^*] \to \mathbb{R}$ and $\phi \in C[\upsilon^*, \vartheta^*]$ with $\upsilon^*, \vartheta^* \in \mathbb{R}$ and $\vartheta^* > 0$. If $\phi \in L^1([\upsilon^*, \vartheta^*])$, then the equality*

$$\frac{\varrho + 2}{\vartheta^* - \upsilon^*}\left\{w(1)\left(\phi(\vartheta^*) + \phi(\upsilon^*)\right) - w(0)\left[\phi\left(\frac{\varrho\vartheta^* + \vartheta^* + \upsilon^*}{\varrho + 2}\right) + \phi\left(\frac{\varrho\upsilon^* + \upsilon^* + \vartheta^*}{\varrho + 2}\right)\right]\right\} -$$
$$- \left(\frac{\varrho + 2}{\vartheta^* - \upsilon^*}\right)^2 \left[J^w_{\frac{\varrho\vartheta^* + \vartheta^* + \upsilon^*}{\varrho+2}+}\phi(\vartheta^*) + J^w_{\frac{\varrho\upsilon^* + \upsilon^* + \vartheta^*}{\varrho+2}-}\phi(\upsilon^*)\right] =$$
$$= \int_0^1 w(\varsigma)\left[\phi'\left(\frac{1+\varrho+\varsigma}{\varrho+2}\vartheta^* + \frac{1-\varsigma}{\varrho+2}\upsilon^*\right) - \phi'\left(\frac{1+\varrho+\varsigma}{\varrho+2}\upsilon^* + \frac{1-\varsigma}{\varrho+2}\vartheta^*\right)\right] d\varsigma$$

is true for every $\varrho \in \mathbb{N}$.

Proof. By properties we have

$$I = \int_0^1 w(\varsigma)\left[\phi'\left(\frac{1+\varrho+\varsigma}{\varrho+2}\vartheta^* + \frac{1-\varsigma}{\varrho+2}\upsilon^*\right) - \phi'\left(\frac{1+\varrho+\varsigma}{\varrho+2}\upsilon^* + \frac{1-\varsigma}{\varrho+2}\vartheta^*\right)\right] d\varsigma =$$
$$= \int_0^1 w(\varsigma)\phi'\left(\frac{1+\varrho+\varsigma}{\varrho+2}\vartheta^* + \frac{1-\varsigma}{\varrho+2}\upsilon^*\right) d\varsigma - \int_0^1 w(\varsigma)\phi'\left(\frac{1+\varrho+\varsigma}{\varrho+2}\upsilon^* + \frac{1-\varsigma}{\varrho+2}\vartheta^*\right) d\varsigma = I_1 - I_2.$$

Integrating by parts and changing the variable in I_1, we state that

$$I_1=\int_0^1 w(\varsigma)\phi'\left(\frac{1+\varrho+\varsigma}{\varrho+2}\vartheta^*+\frac{1-\varsigma}{\varrho+2}\upsilon^*\right)d\varsigma=$$
$$=\frac{\varrho+2}{\vartheta^*-\upsilon^*}\left[w(1)\phi\left(\vartheta^*\right)-w(0)\phi\left(\frac{\varrho\vartheta^*+\vartheta^*+\upsilon^*}{\varrho+2}\right)\right]-$$
$$-\left(\frac{\varrho+2}{\vartheta^*-\upsilon^*}\right)^2\int_{\frac{\varrho\vartheta^*+\vartheta^*+\upsilon^*}{\varrho+2}}^{\vartheta^*} w'\left[\frac{z-\frac{\varrho\vartheta^*+\vartheta^*+\upsilon^*}{\varrho+2}}{\frac{\vartheta^*-\upsilon^*}{\varrho+2}}\right]\phi(z)dz.$$

Since $\frac{\vartheta^*-\upsilon^*}{\varrho+2}=\vartheta^*-\frac{\varrho\vartheta^*+\vartheta^*+\upsilon^*}{\varrho+2}$, finally for I_1 we get

$$I_1=\int_0^1 w(\varsigma)\phi'\left(\frac{1+\varrho+\varsigma}{\varrho+2}\vartheta^*+\frac{1-\varsigma}{\varrho+2}\upsilon^*\right)d\varsigma=$$
$$=\frac{\varrho+2}{\vartheta^*-\upsilon^*}\left[w(1)\phi\left(\vartheta^*\right)-w(0)\phi\left(\frac{\varrho\vartheta^*+\vartheta^*+\upsilon^*}{\varrho+2}\right)\right]-$$
$$-\left(\frac{\varrho+2}{\vartheta^*-\upsilon^*}\right)^2\int_{\frac{\varrho\vartheta^*+\vartheta^*+\upsilon^*}{\varrho+2}}^{\vartheta^*} w'\left[\frac{z-\frac{\varrho\vartheta^*+\vartheta^*+\upsilon^*}{\varrho+2}}{\vartheta^*-\frac{\varrho\vartheta^*+\vartheta^*+\upsilon^*}{\varrho+2}}\right]\phi(z)dz=$$
$$=\frac{\varrho+2}{\vartheta^*-\upsilon^*}\left[w(1)\phi\left(\vartheta^*\right)-w(0)\phi\left(\frac{\varrho\vartheta^*+\vartheta^*+\upsilon^*}{\varrho+2}\right)\right]-\left(\frac{\varrho+2}{\vartheta^*-\upsilon^*}\right)^2 J^w_{\frac{\varrho\vartheta^*+\vartheta^*+\upsilon^*}{\varrho+2}+}\phi\left(\vartheta^*\right). \tag{6}$$

Similarly for I_2, we obtain

$$I_2=\frac{\varrho+2}{a-b}\left[w(1)\phi\left(\upsilon^*\right)-w(0)\phi\left(\frac{\varrho\upsilon^*+\upsilon^*+\vartheta^*}{\varrho+2}\right)\right]+\left(\frac{\varrho+2}{\vartheta^*-\upsilon^*}\right)^2 J^w_{\frac{\varrho\upsilon^*+\upsilon^*+\vartheta^*}{\varrho+2}-}\phi\left(\upsilon^*\right). \tag{7}$$

Subtracting (7) from (6) we obtain the desired equality. This completes the proof. □

Remark 4. Consider in the previous result $\varrho=0$, $w(\varsigma)=\frac{\varsigma^\alpha}{2}-\frac{1}{5}$, then Lemma 2.1 of [16] is easily obtained.

Remark 5. Analogously, if $\varrho=0$, and $w(\varsigma)=\frac{\varsigma^\alpha}{2}-\frac{1}{3}$, then Lemma 1 of [11] is easily obtained.

Remark 6. If we choose $\varrho=0$, and $w(\varsigma)=\frac{\varsigma}{2}-\frac{1}{3}$, then we get Lemma 1 from [10].

Remark 7. In the case $\varrho=0$, $w(\varsigma)=\varsigma$, we obtain Lemma 1 from [5]

$$frac\phi(\upsilon^*)+\phi(\vartheta^*)2-\frac{1}{\vartheta^*-\upsilon^*}\int_{\upsilon^*}^{\vartheta^*}\phi(z)dz=$$
$$=\frac{\vartheta^*-\upsilon^*}{4}\int_0^1\varsigma\left[\phi'\left(\frac{1-\varsigma}{2}\upsilon^*+\frac{1+\varsigma}{2}\vartheta^*\right)-\phi'\left(\frac{1+\varsigma}{2}\upsilon^*+\frac{1-\varsigma}{2}\vartheta^*\right)\right]d\varsigma.$$

Remark 8. Multiplying both sides of (6) and (7) by $\left(\frac{\vartheta^*-\upsilon^*}{\varrho+2}\right)^2$, we get respectively

$$\left(\frac{\vartheta^*-\upsilon^*}{\varrho+2}\right)^2 I_1=\frac{\vartheta^*-\upsilon^*}{\varrho+2}\left[w(1)\phi\left(\vartheta^*\right)-w(0)\phi\left(\frac{\varrho\vartheta^*+\vartheta^*+\upsilon^*}{\varrho+2}\right)\right]-J^w_{\frac{\varrho\vartheta^*+\vartheta^*+\upsilon^*}{\varrho+2}+}\phi\left(\vartheta^*\right)=$$
$$=\left(\frac{\vartheta^*-\upsilon^*}{\varrho+2}\right)^2\int_0^1 w(\varsigma)\phi'\left(\frac{1+\varrho+\varsigma}{\varrho+2}\vartheta^*+\frac{1-\varsigma}{\varrho+2}\upsilon^*\right)d\varsigma,$$
$$\left(\frac{\vartheta^*-\upsilon^*}{\varrho+2}\right)^2 I_2=\frac{\vartheta^*-\upsilon^*}{\varrho+2}\left[w(1)\phi\left(\upsilon^*\right)-w(0)\phi\left(\frac{\varrho\upsilon^*+\upsilon^*+\vartheta^*}{\varrho+2}\right)\right]-J^w_{\frac{\varrho\upsilon^*+\upsilon^*+\vartheta^*}{\varrho+2}-}\phi\left(\upsilon^*\right)=$$
$$=\left(\frac{\vartheta^*-\upsilon^*}{\varrho+2}\right)^2\int_0^1 w(\varsigma)\phi'\left(\frac{1+\varrho+\varsigma}{\varrho+2}\upsilon^*+\frac{1-\varsigma}{\varrho+2}\vartheta^*\right)d\varsigma.$$

Let us call

$$\mathbf{L} = \left(\frac{\vartheta^* - \upsilon^*}{\varrho + 2}\right)^2 I_1 + \left(\frac{\vartheta^* - \upsilon^*}{\varrho + 2}\right)^2 I_2 =$$

$$= \frac{\vartheta^* - \upsilon^*}{\varrho + 2} \left\{ w(1)\left(\phi(\vartheta^*) + \phi(\upsilon^*)\right) - w(0)\left[\phi\left(\frac{\varrho\vartheta^* + \vartheta^* + \upsilon^*}{\varrho + 2}\right) + \phi\left(\frac{\varrho\upsilon^* + \upsilon^* + \vartheta^*}{\varrho + 2}\right)\right]\right\} -$$

$$- \left[J^w_{\frac{\varrho\vartheta^* + \vartheta^* + \upsilon^*}{\varrho + 2}+} \phi\left(\vartheta^*\right) + J^w_{\frac{\varrho\upsilon^* + \upsilon^* + \vartheta^*}{\varrho + 2}-} \phi\left(\upsilon^*\right) \right].$$

Theorem 1. *Let $\phi : [\upsilon^*, \vartheta^*] \to \mathbb{R}$ a differentiable function. If $\phi \in L^1\left([\upsilon^*, \vartheta^*]\right)$, then we have*

$$|\mathbf{L}| \leqslant \mathbf{B} \cdot w(0) \left\|\phi'\right\|_1,$$

where $\mathbf{B} = \left(\frac{\vartheta^* - \upsilon^*}{\varrho + 2}\right)^2$, and $\|\phi'\|_1 = \int_{\upsilon^*}^{\vartheta^*} |\phi'(x)|\, dx < \infty$.

Proof. From Remark 8 after changing variables, we obtain

$$|\mathbf{L}| \leqslant \left(\frac{\vartheta^* - \upsilon^*}{\varrho + 2}\right)^2 \int_0^1 w(\varsigma) \left|\phi'\left(\frac{1 + \varrho + \varsigma}{\varrho + 2}\vartheta^* + \frac{1 - \varsigma}{\varrho + 2}\upsilon^*\right)\right| d\varsigma +$$

$$+ \left(\frac{\vartheta^* - \upsilon^*}{\varrho + 2}\right)^2 \int_0^1 w(\varsigma) \left|\phi'\left(\frac{1 + \varrho + \varsigma}{\varrho + 2}\upsilon^* + \frac{1 - \varsigma}{\varrho + 2}\vartheta^*\right)\right| d\varsigma \leqslant$$

$$\leqslant \left(\frac{\vartheta^* - \upsilon^*}{\varrho + 2}\right)^2 \left\{ \int_{\frac{\varrho\vartheta^* + \vartheta^* + \upsilon^*}{\varrho + 2}}^{\vartheta^*} w'\left[\frac{z - \frac{\varrho\vartheta^* + \vartheta^* + \upsilon^*}{\varrho + 2}}{\frac{\vartheta^* - \upsilon^*}{\varrho + 2}}\right] \left|\phi'(z)\right| dz + \right.$$

$$\left. + \int_{\upsilon^*}^{\frac{\varrho\upsilon^* + \upsilon^* + \vartheta^*}{\varrho + 2}} w'\left[\frac{\frac{\varrho\upsilon^* + \upsilon^* + \vartheta^*}{\varrho + 2} - z}{\frac{\vartheta^* - \upsilon^*}{\varrho + 2}}\right] \left|\phi'(z)\right| dz \right\}.$$

Therefore, the proof is finished. □

Remark 9. If we take $w(\varsigma) = \frac{\varsigma^\alpha}{2} - \frac{1}{5}$ and $\varrho = 0$, we have the Theorem 3.2 of [16].

Theorem 2. *Let $0 < m \leqslant 1$, $0 \leqslant \upsilon^* < \vartheta^*$ and ϕ function defined on the interval $[\upsilon^*, \vartheta^*]$, and $\phi \in C^1(\upsilon^*, \vartheta^*)$, where $\upsilon^*, \vartheta^* \in \mathbb{R}$. If $\phi \in L^1\left([\upsilon^*, \vartheta^*]\right)$ and $|\phi'| \in N^{s,2}_{h,m}[\upsilon^*, \vartheta^*]$ for some fixed $s \in (0, 1]$, then the inequality*

$$\left|\mathbf{A} - \left(\frac{\varrho + 2}{\vartheta^* - \upsilon^*}\right)^2 \left[J^w_{\frac{\varrho\vartheta^* + \vartheta^* + \upsilon^*}{\varrho + 2}+} \phi\left(\vartheta^*\right) + J^w_{\frac{\varrho\upsilon^* + \upsilon^* + \vartheta^*}{\varrho + 2}-} \phi\left(\upsilon^*\right)\right]\right| \leqslant$$

$$\leqslant \left(\left|\phi'(\upsilon^*)\right| + \left|\phi'(\vartheta^*)\right|\right) \int_0^1 w(\varsigma) h^s \left(\frac{1 + \varrho + \varsigma}{\varrho + 2}\right) d\varsigma +$$

$$+ m \left(\left|\phi'\left(\frac{\upsilon^*}{m}\right)\right| + \left|\phi'\left(\frac{\vartheta^*}{m}\right)\right|\right) \int_0^1 w(\varsigma) \left[1 - h\left(\frac{1 + \varrho + \varsigma}{\varrho + 2}\right)\right]^s d\varsigma \tag{8}$$

holds with

$$\mathbf{A} = \frac{\varrho + 2}{\vartheta^* - \upsilon^*} \left\{ w(1)\left(\phi(\vartheta^*) + \phi(\upsilon^*)\right) - w(0)\left[\phi\left(\frac{\varrho\vartheta^* + \vartheta^* + \upsilon^*}{\varrho + 2}\right) + \phi\left(\frac{\varrho\upsilon^* + \upsilon^* + \vartheta^*}{\varrho + 2}\right)\right]\right\}.$$

Proof. By using the second type (h, m)-convex property of the ϕ' function, we get

$$\left|\phi'\left(\frac{1 + \varrho + \varsigma}{\varrho + 2}\vartheta^* + \frac{1 - \varsigma}{\varrho + 2}\upsilon^*\right)\right| = \left|\phi'\left(\left(\frac{1 + \varrho + \varsigma}{\varrho + 2}\right)\vartheta^* + \left(1 - \frac{1 + \varrho + \varsigma}{\varrho + 2}\right)\upsilon^*\right)\right| \leqslant$$

$$\leqslant h^s\left(\frac{1+\varrho+\varsigma}{\varrho+2}\right)|\phi'(\vartheta^*)|+m\left[1-h\left(\frac{1+\varrho+\varsigma}{\varrho+2}\right)\right]^s\left|\phi'(\frac{\upsilon^*}{m})\right|$$

and

$$\left|\phi'\left(\frac{1+\varrho+\varsigma}{\varrho+2}\upsilon^*+\frac{1-\varsigma}{\varrho+2}\vartheta^*\right)\right|\leqslant h^s\left(\frac{1+\varrho+\varsigma}{\varrho+2}\right)|\phi'(\upsilon^*)|+m\left[1-h\left(\frac{1+\varrho+\varsigma}{\varrho+2}\right)\right]^s\left|\phi'(\frac{\vartheta^*}{m})\right|.$$

So, using the Lemma 1 we obtain

$$\left|\mathbf{A}-\left(\frac{\varrho+2}{\vartheta^*-\upsilon^*}\right)^2\left[J^w_{\frac{\varrho\vartheta^*+\vartheta^*+\upsilon^*}{\varrho+2}+}\phi(\vartheta^*)+J^w_{\frac{\varrho\upsilon^*+\upsilon^*+\vartheta^*}{\varrho+2}-}\phi(\upsilon^*)\right]\right|\leqslant$$
$$\leqslant(\phi'(\upsilon^*)+\phi'(\vartheta^*))\int_0^1 w(\varsigma)\left[1-h\left(\frac{1+\varrho+\varsigma}{\varrho+2}\right)\right]^s d\varsigma+$$
$$+m\left(\phi'(\frac{\upsilon^*}{m})+\phi'(\frac{\vartheta^*}{m})\right)\int_0^1 w(\varsigma)h^s\left(\frac{1+\varrho+\varsigma}{\varrho+2}\right)d\varsigma.$$

Which is the inequality sought, this ends the proof. □

Remark 10. Taking ϕ' s-convex function, $\varrho=0$, and $w(\varsigma)=\frac{\varsigma^\alpha}{2}-\frac{1}{3}$, from (8) we obtain the Theorem 6 of [11]. Under these conditions, Corollary 1 and Remark 3 of this paper remain valid.

Remark 11. If we take $|\phi'|$ s-convex, i.e. $h(\vartheta)=\vartheta$, $m=1$ and $w(\varsigma)=\varsigma^\alpha$, and $\varrho=0$, we get Theorem 5 of [30].

Theorem 3. *Let $0<m\leqslant 1$, $0\leqslant\upsilon^*<\vartheta^*$ and ϕ function defined on the interval $[\upsilon^*,\vartheta^*]$, and $\phi\in C^1(\upsilon^*,\vartheta^*)$, where $\upsilon^*,\vartheta^*\in\mathbb{R}$. If $\phi\in L^1([\upsilon^*,\vartheta^*])$ and $|\phi'|^q\in N^{s,2}_{h,m}[\upsilon^*,\vartheta^*]$ for some fixed $s\in(0,1]$ and $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then the following inequality is fulfilled*

$$\left|\mathbf{A}-\left(\frac{\varrho+2}{\vartheta^*-\upsilon^*}\right)^2\left[J^w_{\frac{\varrho\vartheta^*+\vartheta^*+\upsilon^*}{\varrho+2}+}\phi(\vartheta^*)+J^w_{\frac{\varrho\upsilon^*+\upsilon^*+\vartheta^*}{\varrho+2}-}\phi(\upsilon^*)\right]\right|\leqslant$$
$$\leqslant\mathbf{B}\left[\left(|\phi'(\upsilon^*)|^q+|\phi'(\vartheta^*)|^q\right)\mathbf{C}+m\left(\left|\phi'\left(\frac{\upsilon^*}{m}\right)\right|^q+\left|\phi'\left(\frac{\vartheta^*}{m}\right)\right|^q\right)\mathbf{D}\right],\tag{9}$$

with $\mathbf{A}$ *as before, and* $\mathbf{B}$, $\mathbf{C}$ *and* $\mathbf{D}$ *they will be specified later.*

Proof. As in the proof of the previous result, we have

$$\left|\mathbf{A}-\left(\frac{\varrho+2}{\vartheta^*-\upsilon^*}\right)^2\left[J^w_{\frac{\varrho\vartheta^*+\vartheta^*+\upsilon^*}{\varrho+2}+}\phi(\vartheta^*)+J^w_{\frac{\varrho\upsilon^*+\upsilon^*+\vartheta^*}{\varrho+2}-}\phi(\upsilon^*)\right]\right|\leqslant$$
$$\leqslant\int_0^1 w(\varsigma)\left|\phi'\left(\frac{1+\varrho+\varsigma}{\varrho+2}\vartheta^*+\frac{1-\varsigma}{\varrho+2}\upsilon^*\right)\right|d\varsigma+\int_0^1 w(\varsigma)\left|\phi'\left(\frac{1+\varrho+\varsigma}{\varrho+2}\upsilon^*+\frac{1-\varsigma}{\varrho+2}\vartheta^*\right)\right|d\varsigma.\tag{10}$$

Using the Hölder inequality on the two integrals of (10) gives us $\left(\frac{1}{p}+\frac{1}{q}=1\right)$:

$$\int_0^1 w(\varsigma)\left|\phi'\left(\frac{1+\varrho+\varsigma}{\varrho+2}\vartheta^*+\frac{1-\varsigma}{\varrho+2}\upsilon^*\right)\right|d\varsigma\leqslant$$
$$\leqslant\left(\int_0^1 w^p(\varsigma)d\varsigma\right)^{\frac{1}{p}}\left[\int_0^1\left|\phi'\left(\frac{1+\varrho+\varsigma}{\varrho+2}\vartheta^*+\frac{1-\varsigma}{\varrho+2}\upsilon^*\right)\right|^q d\varsigma\right]^{\frac{1}{q}},\tag{11}$$
$$\int_0^1 w(\varsigma)\left|\phi'\left(\frac{1+\varrho+\varsigma}{\varrho+2}\vartheta^*+\frac{1-\varsigma}{\varrho+2}\upsilon^*\right)\right|d\varsigma\leqslant$$

$$\leqslant \left(\int_0^1 w^p(\varsigma)d\varsigma\right)^{\frac{1}{p}} \left[\int_0^1 \left|\phi'\left(\frac{1+\varrho+\varsigma}{\varrho+2}\upsilon^* + \frac{1-\varsigma}{\varrho+2}\vartheta^*\right)\right|^q d\varsigma\right]^{\frac{1}{q}}. \tag{12}$$

Taking into account the (h,m)-convexity of $|\phi'|^q$ we obtain

$$\left[\int_0^1 \left|\phi'\left(\frac{1+\varrho+\varsigma}{\varrho+2}\vartheta^* + \frac{1-\varsigma}{\varrho+2}\upsilon^*\right)\right|^q d\varsigma\right]^{\frac{1}{q}} \leqslant$$

$$\leqslant \left[|\phi'(\vartheta^*)|^q \int_0^1 h^s\left(\frac{1+\varrho+\varsigma}{\varrho+2}\right)d\varsigma + m\left|\phi'(\frac{\upsilon^*}{m})\right|^q \int_0^1 \left(1-h\left(\frac{1+\varrho+\varsigma}{\varrho+2}\right)\right)^s d\varsigma\right]^{\frac{1}{q}}, \tag{13}$$

$$\left[\int_0^1 \left|\phi'\left(\frac{1+\varrho+\varsigma}{\varrho+2}\upsilon^* + \frac{1-\varsigma}{\varrho+2}\vartheta^*\right)\right|^q d\varsigma\right]^{\frac{1}{q}} \leqslant$$

$$\leqslant \left[|\phi'(\upsilon^*)|^q \int_0^1 h^s\left(\frac{1+\varrho+\varsigma}{\varrho+2}\right)d\varsigma + m\left|\phi'(\frac{\vartheta^*}{m})\right|^q \int_0^1 \left(1-h\left(\frac{1+\varrho+\varsigma}{\varrho+2}\right)\right)^s d\varsigma\right]^{\frac{1}{q}}, \tag{14}$$

so, using (13) and (14) in (11) and (12), and then in (10), rearranging, grouping and denoting $\mathbf{B} = \left(\int_0^1 w^p(\varsigma)d\varsigma\right)^{\frac{1}{p}}$, $\mathbf{C} = \int_0^1 h^s\left(\frac{1+\varrho+\varsigma}{\varrho+2}\right)d\varsigma$ and $\mathbf{D} = \int_0^1 \left(1-h\left(\frac{1+\varrho+\varsigma}{\varrho+2}\right)\right)^s d\varsigma$ leads us to the desired inequality. □

Remark 12. As in the previous Remark, this result yields Theorem 6 and Corollary 1 of [30].

Remark 13. If we work with convex functions, i.e. $h(\varsigma)=\varsigma$, $s=m=1$ and $w(\varsigma)=\varsigma$, then the above result becomes Theorem 2.3 from [31]. Theorem 1 of [32] is also a particular case of this result.

Remark 14. The Theorem 7, Corollary 2, and Remark 4 of [11] are also particular cases of this result.

The next result is a different version of (9).

Theorem 4. *Let $0 < m \leqslant 1$, $0 \leqslant \upsilon^* < \vartheta^*$ and ϕ function defined on the interval $[\upsilon^*, \vartheta^*]$, and $\phi \in C^1(\upsilon^*, \vartheta^*)$, where $\upsilon^*, \vartheta^* \in \mathbb{R}$. If $\phi \in L^1([\upsilon^*, \vartheta^*])$ and $|\phi'|^q \in N_{h,m}^{s,2}[\upsilon^*, \vartheta^*]$ for some fixed $s \in (0,1]$ and $q > 1$, then the following inequality is fulfilled*

$$\left|\mathbf{A} - \left(\frac{\varrho+2}{\vartheta^*-\upsilon^*}\right)^2 \left[J^w_{\frac{\varrho\vartheta^*+\vartheta^*+\upsilon^*}{\varrho+2}+}\phi(\vartheta^*) + J^w_{\frac{\varrho\upsilon^*+\upsilon^*+\vartheta^*}{\varrho+2}-}\phi(\upsilon^*)\right]\right| \leqslant$$

$$\leqslant \mathbf{E}\left[\left(|\phi'(\upsilon^*)|^q + |\phi'(\vartheta^*)|^q\right)\mathbf{F} + m\left(\left|\phi'(\frac{\upsilon^*}{m})\right|^q + \left|\phi'(\frac{\vartheta^*}{m})\right|^q\right)\mathbf{G}\right],$$

with $\mathbf{A}$ as before, $\mathbf{E} = \left(\int_0^1 w(\varsigma)d\varsigma\right)^{1-\frac{1}{p}}$, $\mathbf{F} = \int_0^1 w(\varsigma)h^s\left(\frac{1+\varrho+\varsigma}{\varrho+2}\right)d\varsigma$ and

$$\mathbf{G} = \int_0^1 w(\varsigma)\left(1-h\left(\frac{1+\varrho+\varsigma}{\varrho+2}\right)\right)^s d\varsigma.$$

Proof. As before

$$\left|\mathbf{A} - \left(\frac{\varrho+2}{\vartheta^*-\upsilon^*}\right)^2 \left[J^w_{\frac{\varrho\vartheta^*+\vartheta^*+\upsilon^*}{\varrho+2}+}\phi(\vartheta^*) + J^w_{\frac{\varrho\upsilon^*+\upsilon^*+\vartheta^*}{\varrho+2}-}\phi(\upsilon^*)\right]\right| \leqslant$$

$$\leqslant \int_0^1 w(\varsigma)\left|\phi'\left(\frac{1+\varrho+\varsigma}{\varrho+2}\vartheta^* + \frac{1-\varsigma}{\varrho+2}\upsilon^*\right)\right|d\varsigma + \int_0^1 w(\varsigma)\left|\phi'\left(\frac{1+\varrho+\varsigma}{\varrho+2}\upsilon^* + \frac{1-\varsigma}{\varrho+2}\vartheta^*\right)\right|d\varsigma.$$

Using now that well-known power mean inequality with modulus properties takes us to:

$$\int_0^1 w(\varsigma)\left|\phi'\left(\frac{1+\varrho+\varsigma}{\varrho+2}\vartheta^*+\frac{1-\varsigma}{\varrho+2}\upsilon^*\right)\right|d\varsigma+\int_0^1 w(\varsigma)\left|\phi'\left(\frac{1+\varrho+\varsigma}{\varrho+2}\upsilon^*+\frac{1-\varsigma}{\varrho+2}\vartheta^*\right)\right|d\varsigma\leqslant$$
$$\leqslant\left(\int_0^1 w(\varsigma)d\varsigma\right)^{1-\frac{1}{q}}\left[\int_0^1 w(\varsigma)\left|\phi'\left(\frac{1+\varrho+\varsigma}{\varrho+2}\vartheta^*+\frac{1-\varsigma}{\varrho+2}\upsilon^*\right)\right|d\varsigma\right]^{\frac{1}{q}}+$$
$$+\left(\int_0^1 w(\varsigma)d\varsigma\right)^{1-\frac{1}{q}}\left[\int_0^1 w(\varsigma)\left|\phi'\left(\frac{1+\varrho+\varsigma}{\varrho+2}\upsilon^*+\frac{1-\varsigma}{\varrho+2}\vartheta^*\right)\right|d\varsigma\right]^{\frac{1}{q}}.$$

From the (h, m)-convexity of $|\phi'|^q$ and a simple but tedious algebraic work, the proof of the Theorem is completed. ☐

Remark 15. Under assumption $|\phi'|^q$ s-convex and $w(\varsigma)=\varsigma^\alpha$, we get Theorem 7 of [30]. If we additionally put that $\alpha = 1$, it follows Theorem 2 of [32] and Theorem 1 of [33]. The reader can also easily check that Theorem 9 and Corollary 4 of [11] are particular cases of our result and that Remark 6 of said work is still valid.

Conclusions

In this paper, various extensions and generalizations of the classical Simpson's inequality have been established, in the context of weighted integral operators. Throughout our work, we have seen how various results reported in the literature are particular cases of ours, which shows the breadth of strength of these. However, we did not want to conclude without pointing out two more aspects regarding the breadth of our results. Firstly, referring to the integral operator used, given that the weight function can include several known cases, we can add that if $w'(\varsigma)=\frac{(\sigma-\upsilon^*)^{\alpha-1}\varsigma^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}$ (or that is, we consider the k-integral of [34]), the Lemma 1 reduces to Lemma 2.1 of [35], obviously many of the results of that work, can also be obtained from ours, considering convex functions. The second issue is the notion of convexity used, which we have shown contains several well-known. This, together with the way of writing the argument (in reality we obtain families of inequalities), means that our results cover many of those published so far. Finally, we want to point out that this weighted operator can be used in the study of other inequalities, for example, the Minkowski inequality (see, for example, [36]), in this paper they use the weight indicated above, thus these results can be generalized using a general weight $w'(\varsigma)$.

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