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Article

# Global solvability of the inverse spectral problem for differential systems on a finite interval

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**Abstract.** The inverse spectral problem is studied for non-selfadjoint systems of ordinary differential equations on a finite interval. We provide necessary and sufficient conditions for the global solvability of the inverse problem, along with an algorithm for constructing its solution. For solving this nonlinear inverse problem, we develop ideas of the method of spectral mappings, which allows one to construct the potential matrix from the given spectral characteristics and establish conditions for the global solvability of the inverse problem considered.

**Keywords:** differential systems, spectral characteristics, inverse problems, method of spectral mappings, global solvability

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# Глобальная разрешимость обратной спектральной задачи для дифференциальных систем на конечном интервале

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**Аннотация.** Исследуется обратная спектральная задача для несамосопряженных систем обыкновенных дифференциальных уравнений на конечном интервале. Получены необходимые и достаточные условия глобальной разрешимости обратной задачи, а также алгоритм построения ее решения. Для решения этой нелинейной обратной задачи используется развитие идей метода спектральных отображений, что позволяет построить потенциальную матрицу по заданным спектральным характеристикам и установить условия глобальной разрешимости рассматриваемой обратной задачи.

**Ключевые слова:** дифференциальные системы, спектральные характеристики, обратные задачи, метод спектральных отображений, глобальная разрешимость

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## Introduction

We study the inverse spectral problem for the following system of differential equations on a finite interval

$$\ell Y(x) := Q_0 Y'(x) + Q(x)Y(x) = \rho Y(x), \quad 0 \leqslant x \leqslant T. \tag{1}$$

Here $Y = [y_k]^t_{k=\overline{1,n}}$ is a column-vector ($t$ denotes the transposition), $\rho$ is the spectral parameter, $Q_0 = \operatorname{diag}[q_k]_{k=\overline{1,n}}$, where $q_k \neq 0$ are different complex numbers, and $Q(x) = [q_{kj}(x)]_{k,j=\overline{1,n}}$, where $q_{kj}(x) \in L(0,T)$ are complex-valued functions, and $q_{kk}(x) \equiv 0$. The matrix $Q(x)$ is called the potential.

Inverse problems for systems of the form (1) in different formulations were studied by many authors. Most of the works are devoted to inverse problems for Dirac and AKNS systems, i.e. to the case of $n = 2$ (see, for example, [1]). The main tool for these investigations was the transformation operator method, and the obtained results are similar to the results for the Sturm-Liouville operator (see [1, 2] and the references therein). These results can be easily generalized by the same method for $n > 2$ but only in the exceptional case when all $q_k$ are real (see, for example, [3–5]). However, for system (1) of general form with *arbitrary complex* $q_k$ and arbitrary integrable potentials the inverse problem is essentially more difficult to study. The method of spectral mappings [6] was created for solving the inverse problem in this general case. In particular, by this method the inverse problem for system (1) on the half-line has been solved in [7]. The inverse problem for system (1) on a finite interval in the general case has been solved in [8], where the uniqueness theorem was proved and a constructive procedure for the solution of the inverse problem was suggested.

In this paper, we study the inverse spectral problem for system (1) with arbitrary complex $q_k$ on a finite interval. As the main spectral characteristic we introduce and investigate the so-called Weyl matrix which is an analog of the classical Weyl function for the Sturm – Liouville operator and the Weyl matrix introduced in [9, 10] for higher-order differential operators. The main method of the investigation is the method of spectral mappings. Developing the ideas of this method in connection with systems of the form (1) and using discovered properties of the Weyl matrix, we provide necessary and sufficient conditions for the solvability of the inverse problem considered. We will use some notations and facts from [8].

## 1. Properties of the spectral characteristics

Let the matrices $h = [h_{\xi\nu}]_{\xi,\nu=\overline{1,n}}$ and $H = [H_{\xi\nu}]_{\xi,\nu=\overline{1,n}}$ be given, where $h_{\xi\nu}$, $H_{\xi\nu}$ are complex numbers, and $\det h \neq 0$, $\det H \neq 0$. We introduce the linear forms $U(Y) = [U_\xi(Y)]^t_{\xi=\overline{1,n}}$, $V(Y) = [V_\xi(Y)]^t_{\xi=\overline{1,n}}$ by the formulae $U(Y) = hY(0)$, $V(Y) = HY(T)$, i.e.

$$U_\xi(Y) = [h_{\xi 1}, \dots, h_{\xi n}]Y(0), \quad V_\xi(Y) = [H_{\xi 1}, \dots, H_{\xi n}]Y(T).$$

Let the vector-functions $\Phi_m(x,\rho) = [\Phi_{\nu m}(x,\rho)]^t_{\nu=\overline{1,n}}$, $m = \overline{1,n}$ be solutions of system (1) under the conditions $U_\xi(\Phi_m) = \delta_{\xi m}$, $\xi = \overline{1,m}$, $V_\eta(\Phi_m) = 0$, $\eta = \overline{1,n-m}$. Here and in the sequel, $\delta_{\xi m}$ is the Kronecker delta. Let $M_{m\xi}(\rho) = U_\xi(\Phi_m)$, $M(\rho) = [M_{m\xi}(\rho)]_{m,\xi=\overline{1,n}}$, $\Phi(x,\rho) = [\Phi_1(x,\rho), \dots, \Phi_n(x,\rho)] = [\Phi_{\nu m}(x,\rho)]_{\nu,m=\overline{1,n}}$. The functions $M_{\xi m}(\rho)$ are called the *Weyl functions*. The matrix $M(\rho)$ is called the *Weyl matrix* for system (1).

**Inverse problem 1.** *Fix $Q_0, h, H$, i.e. the numbers $q_k, h_{\xi\nu}, H_{\xi\nu}$, $k, \xi, \nu = \overline{1,n}$ are known and fixed. Given the Weyl matrix $M(\rho)$, construct the potential $Q(x)$, $0 < x < T$.*

It is easy to see that in the general case, the specification of the Weyl matrix does not determine uniquely the matrices of the linear forms $h$ and $H$. It is possible to point out particular cases (when the matrices $h$ and $H$ are normalized and have a special structure) for which the specification of the Weyl matrix uniquely determines not only the potential but also the matrices of the linear forms. For simplicity, we confine ourselves here to the most principal and difficult problem of recovering the potential $Q(x)$.

Let $C(x,\rho) = [C_{km}(x,\rho)]_{k,m=\overline{1,n}}$ be a matrix-solution of system (1) under the initial condition $U(C) = hC(0,\rho) = I$ (here and below $I$ denotes the identity matrix of the corresponding dimension or the identity operator in the corresponding Banach space). In other words, the column-vectors $C_m(x,\rho) = [C_{km}(x,\rho)]^t_{k=\overline{1,n}}$, $m = \overline{1,n}$, are solutions of (1) under the initial conditions $U_\xi(C_m) = \delta_{\xi m}$, $\xi, m = \overline{1,n}$. The functions $C_{km}(x,\rho)$ are entire in $\rho$ for each fixed $x$. Clearly, $\Phi(x,\rho) = C(x,\rho)\mathcal{N}(\rho)$, where $\mathcal{N}(\rho) := M^t(\rho)$. One has

$$M_{mk}(\rho) = (\Delta_{mm}(\rho))^{-1}\Delta_{mk}(\rho), \quad 1 \leqslant m < k \leqslant n, \tag{2}$$

where

$$\Delta_{mk}(\rho) := (-1)^{m+k}\det[V_\xi(C_\nu)]_{\xi=\overline{1,n-m};\nu=\overline{m,n}\setminus k}, \quad 1 \leqslant m \leqslant k \leqslant n, \quad \Delta_{nn}(\rho) := 1.$$

It is known that the $\rho$-plane can be partitioned into sectors $S_j = \{\rho: \ \arg\rho \in (\theta_j, \theta_{j+1})\}$, $j = \overline{0, 2r-1}$, $0 \leqslant \theta_0 < \theta_1 < \ldots < \theta_{2r-1} \leqslant 2\pi$ in which there exist permutations $i_k = i_k(S_j)$ of the numbers $1, \ldots, n$, such that for the numbers $R_k = R_k(S_j)$ of the form $R_k = \beta_{i_k}$ one has

$$\mathrm{Re}(\rho R_1) < \ldots < \mathrm{Re}(\rho R_n), \quad \rho \in S_j. \tag{3}$$

We put $\theta_{j+2kr} := \theta_j$, $S_{j+2kr} := S_j$, $k \in \mathbf{Z}$, and denote $\Gamma_j = \{\rho: \ \arg\rho = \theta_j\}$. Clearly, $\Gamma_{j+2kr} := \Gamma_j$, $k \in \mathbf{Z}$. We note that the number $2r$ of the sectors $S_j$ depends on the location of the numbers $\{\beta_k\}_{k=\overline{1,n}}$ on the complex-plane, and $1 \leqslant r \leqslant n(n-1)/2$. For example, if all $\beta_k$ lie on a line containing the origin, then $r = 1$. Denote

$$\Omega^0_{mk}(j_1, \ldots, j_m) := \det[h_{\xi, j_\nu}]_{\xi=\overline{1,m-1},k;\ \nu=\overline{1,m}}, \quad 1 \leqslant m \leqslant k \leqslant n,$$
$$\Omega^0_m(j_1, \ldots, j_m) := \Omega^0_{mm}(j_1, \ldots, j_m), \quad \Omega^0_0 := 1,$$
$$\Omega^1_m(j_{m+1}, \ldots, j_n) := \det[H_{\xi, j_\nu}]_{\xi=\overline{1,n-m}, \quad \nu=\overline{m+1,n}}.$$

Let

$$\Omega^0_m(i_1, \ldots, i_m) \neq 0, \quad \Omega^1_m(i_{m+1}, \ldots, i_n) \neq 0, \quad m = \overline{1, n-1}, \quad j = \overline{0, 2r-1}, \tag{4}$$

where $i_k = i_k(S_j)$ is the above-mentioned perturbation for the sector $S_j$. Condition (4) is called the *regularity condition*. Systems, that do not satisfy the regularity condition, possess qualitatively different properties for investigating inverse problems and are not considered in this paper.

Fix $j = \overline{0, 2r-1}$. For $\rho \in \Gamma_j$, strict inequalities from (3) in some places become equalities. Let $m_i = m_i(j)$, $p_i = p_i(j)$, $i = \overline{1,s}$, be such that for $\rho \in \Gamma_j$,

$$\mathrm{Re}(\rho R_{m_i - 1}) < \mathrm{Re}(\rho R_{m_i}) = \ldots = \mathrm{Re}(\rho R_{m_i + p_i}) < \mathrm{Re}(\rho R_{m_i + p_i + 1}), \quad i = \overline{1,s},$$

where $R_k = R_k(S_j)$.

Denote $N_j := \{m: \ m = \overline{m_1, m_1 + p_1 - 1}, \ldots, \overline{m_s, m_s + p_s - 1}\}$, $J_m := \{j: \ m \in N_j\}$, $\gamma_m = \bigcup\limits_{j \in J_m} \Gamma_j$, $\Sigma_m = \mathbf{C} \setminus \gamma_m$ is the $\rho$- plane without the cuts along the rays from $\gamma_m$. Clearly, the domain $\Sigma_m = \bigcup\limits_\nu S_{m\nu}$ consists of sectors $S_{m\nu}$, each of which is a union of several sectors $S_j$ with the same collection $\{R_\xi\}_{\xi=\overline{1,m}}$. Let $\Gamma_{j,\sigma} := \{\rho: \ \mathrm{dist}\,(\rho, \Gamma_j) \leqslant \sigma\}$, $\sigma > 0$, be a strip along the ray $\Gamma_j$, and let $\gamma_{m,\sigma} := \bigcup\limits_{j \in J_m} \Gamma_{j,\sigma} = \{\rho: \ \mathrm{dist}\,(\rho, \gamma_m) \leqslant \sigma\}$. Denote $\Gamma^1_\sigma := \bigcup\limits_{j=0}^{2r-1} \Gamma_{j,\sigma}$, $\Gamma^1 := \bigcup\limits_{j=0}^{2r-1} \Gamma_j$.

Denote by $\Lambda_m = \{\rho_{lm}\}_{l\geqslant 1}$, $m = \overline{1, n-1}$, the set of zeros (with multiplicities) of the entire function $\Delta_{mm}(\rho)$, and put $\Lambda := \bigcup\limits_{m=1}^{n-1} \Lambda_m$. The numbers $\{\rho_{lm}\}_{l\geqslant 1}$ coincide with the eigenvalues of the boundary value problem $L_m$ for system (1) under the conditions $U_\xi(Y) = V_\eta(y) = 0$, $\xi = \overline{1,m}$, $\eta = \overline{1, n-m}$. Denote $G_{\delta,m} = \{\rho : \ |\rho - \rho_{lm}| \geqslant \delta, \ l \geqslant 1\}$, $G_\delta = \bigcap\limits_{m=1}^{n-1} G_{\delta,m}$. Let $\Lambda_m^0 = \{\rho_{lm}^0\}_{l\geqslant 1}$, $m = \overline{1, n-1}$, be the eigenvalues of the "simplest" boundary value problems $L_m^0$ for system (1) with $Q(x) \equiv 0$. The following properties of the Weyl matrix were established in [8].

**Theorem 1.** *1. The Weyl functions $M_{mk}(\rho)$, $k > m$ are meromorphic in $\rho$ with the set $\Lambda_m = \{\rho_{lm}\}_{l\geqslant 1}$ of poles. For $|\rho| \to \infty$, $\arg\rho = \theta \in (\theta_j, \theta_{j+1})$,*

$$M_{mk}(\rho) = M_{mk}^0 + O(\rho^{-1}),$$

*where $M_{mk}^0 = (\Omega_m^0(i_1, \dots, i_m))^{-1}\Omega_{mk}^0(i_1, \dots, i_m)$. Moreover,*

$$|M_{mk}(\rho)| \leqslant C_\delta, \quad \rho \in G_{\delta,m}\,. \tag{5}$$

*2. There exists $\sigma > 0$ such that $\Lambda_m \subset \gamma_{m,\sigma}$. Moreover, $\Lambda_m = \bigcup\limits_{j\in J_m} \Lambda_{mj}$, where $\Lambda_{mj} \in \Gamma_{j,\sigma}$ is the subsequence of $\Lambda_m$ located in the strip $\Gamma_{j,\sigma}$, $j \in J_m$.*

*3. The number $n_{ma}$ of zeros of $\Delta_{mm}(\rho)$ in the domain $\gamma_{m,\sigma}^a := \{\rho : \ \rho \in \gamma_{m,\sigma}, |\rho| \in [a, a+1]\}$ is bounded with respect to $a$.*

*4. There exist positive numbers $r_N \to \infty$ such that for sufficiently small $\delta > 0$, the circles $|\rho| = r_N$ lie in $G_\delta$ for all $N$.*

*5. For $m = \overline{1, n-1}$, one has $\rho_{lm} = \rho_{lm}^0 + O(l^{-1})$ as $l \to \infty$.*

We consider the differential system

$$\ell^* Z(x) := -Z'(x)Q_0 + Z(x)Q(x) = \rho Z(x),$$

where $Z = [z_k]_{k=\overline{1,n}}$ is a row-vector. Clearly,

$$Z(x)\ell Y(x) - \ell^* Z(x) Y(x) = \frac{d}{dx}\Big(Z(x)Q_0Y(x)\Big). \tag{6}$$

It follows from (6) that if $\ell Y(x,\rho) = \rho Y(x,\rho)$ and $\ell^* Z(x,\mu) = \mu Z(x,\mu)$, then

$$(\rho - \mu) Z(x,\mu) Y(x,\rho) = \frac{d}{dx}\Big(Z(x,\mu)Q_0Y(x,\rho)\Big). \tag{7}$$

Put $U^*(Z) = Z(0)h^*$, $V^*(Z) = Z(T)H^*$, where $h^* = [h_{k\xi}^*]_{k,\xi=\overline{1,n}} := Q_0h^{-1}$, $H^* = [H_{k\xi}^*]_{k,\xi=\overline{1,n}} := Q_0H^{-1}$. Then $U^*(Z) = [U_n^*(Z), \dots, U_1^*(Z)]$, $V^*(Z) = [V_n^*(Z), \dots, V_1^*(Z)]$, where $U_{n-\xi+1}^*(Z) = Z(0)[h_{k\xi}^*]_{k=\overline{1,n}}^t$, $V_{n-\xi+1}^*(Z) = Z(T)[H_{k\xi}^*]_{k=\overline{1,n}}^t$

Denote $R_m^* := -R_{n-m+1}$. Let vector-functions $\Phi_m^*(x,\rho) = [\Phi_{km}^*(x,\rho)]_{k=\overline{1,n}}$, $m = \overline{1,n}$ be solutions of the equation $\ell^* Z = \rho Z$, satisfying the conditions $U_\xi^*(\Phi_m^*) = \delta_{\xi m}$, $\xi = \overline{1,m}$, $V_\eta^*(\Phi_m^*) = 0$, $\eta = \overline{1, n-m}$. We put $\Phi^*(x,\rho) = [\Phi_{n-m+1}^*(x,\rho)]_{m=\overline{1,n}}^t = [\Phi_{n-m+1,k}^*(x,\rho)]_{m,k=\overline{1,n}}$.

## 2. Solution of the inverse problem

It was proved in [8] that *the specification of the Weyl matrix $M(\rho)$ uniquely determines the potential matrix $Q(x)$*. In this section, we provide a constructive solution to the inverse problem of recovering the potential matrix $Q(x)$ from the given Weyl matrix $M(\rho)$. For this purpose, we reduce our nonlinear inverse problem to the solution of the so-called *main equation*, which is a linear equation in a corresponding Banach space of sequences. We give a derivation of the main equation and prove its unique solvability. Using the solution of the main equation we provide an algorithm for the solution of the inverse problem along with necessary and sufficient conditions for its solvability. For simplicity, in the sequel we confine ourselves to the case when the functions $\Delta_{mm}(\rho)$, $m=\overline{1,n-1}$, have only simple zeros (the general case requires minor technical modifications).

For $\rho_0 \in \Lambda$ we define the matrix $F(\rho_0)=[F_{jk}(\rho_0)]_{j,k=\overline{1,n}}$ via

$$F(\rho_0)=-(N_{\langle 0\rangle}(\rho_0))^{-1}N_{\langle -1\rangle}(\rho_0).$$

Here and below $f_{\langle k\rangle}(\rho_0)=(f(\rho))^{\langle k\rangle}_{|\rho=\rho_0}$ denotes the $k$-th Laurent's coefficient for the function $f(\rho)$ at the point $\rho=\rho_0$. In particular, $f_{\langle -1\rangle}(\rho_0)=\operatorname*{Res}_{\rho=\rho_0} f(\rho)$.

Denote $A_s(\rho_0)=[F_{\nu j}(\rho_0)]_{j=\overline{1,n-s},\,\nu=\overline{n-s,n}}$, $s=\overline{1,n-1}$, $\Lambda_0=\Lambda_n=\varnothing$.

**Property** $S_1$**.** *If $\rho_0\notin\Lambda_m$, then $F_{m+1,j}(\rho_0)=\ldots=F_{nj}(\rho_0)=0$, $j=\overline{1,m}$. If, moreover, $\nu<m-1$, $\rho_0\notin\Lambda_\nu$, $\rho_0\in\Lambda_{\nu+1}\bigcap\ldots\bigcap\Lambda_{m-1}$, $\rho_0\notin\Lambda_m$, $1\leqslant\nu+1<m\leqslant n$, then $F_{m,\nu+1}(\rho_0)\neq 0$.*

**Property** $S_2$**.** *The following relation holds rank $A_s(\rho_0)\leqslant 1$, $s=\overline{1,n-1}$.*

Fix $N\geqslant 1$. Denote by $W_N$ the set of functions $f(x)$ such that $f^{(\nu)}(x)$, $\nu=\overline{0,N-1}$, are absolutely continuous on $[0,T]$. We will write $\ell\in V_N$ if $Q(x)\in W_N$. We will solve the inverse problem in the classes $V_N$.

Let $\ell\in V_N$ and let the Weyl matrix $M(\rho)$ for system (1) be given. We take an arbitrary off-diagonal potential $\tilde Q(x)\in W_N$. We agree that everywhere below if a symbol $\alpha$ denotes an object related to $Q$, then $\tilde\alpha$ will denote the analogous object related to $\tilde Q$, and $\hat\alpha:=\alpha-\tilde\alpha$.

Choose $\sigma>0$ such that $\Lambda^1:=\Lambda\cup\tilde\Lambda\subset\Gamma^1_\sigma$. Let $\omega$ be the contour (with a counterclockwise circuit) which is the boundary of $\Gamma^1_\sigma$, i.e. $\omega:=\{\rho:\ \operatorname{dist}(\rho,\Gamma^1)=\sigma\}$.

Denote $J:=\{\rho:\ \operatorname{dist}(\rho,\Gamma^1)>\sigma\}$. Then the following relations hold

$$\tilde\Phi(x,\rho)=\Phi(x,\rho)+\frac{1}{2\pi i}\int_\omega \Phi(x,\mu)\tilde\Phi^*(x,\mu)Q_0\tilde\Phi(x,\rho)\frac{d\mu}{\mu-\rho},\quad \rho\in J, \tag{8}$$

$$\frac{\Phi^*(x,\theta)Q_0\Phi(x,\rho)}{\rho-\theta}-\frac{\tilde\Phi^*(x,\theta)Q_0\tilde\Phi(x,\rho)}{\rho-\theta}=$$

$$=\frac{1}{2\pi i}\int_\omega \Phi^*(x,\theta)Q_0\Phi(x,\mu)\tilde\Phi^*(x,\mu)Q_0\tilde\Phi(x,\rho)\,\frac{d\mu}{(\mu-\theta)(\rho-\mu)}\,,\quad \rho,\theta\in J. \tag{9}$$

Denote

$$\xi_l:=\sum_{k=2}^{n}\Big(|\rho_{l,k-1}-\tilde\rho_{l,k-1}|+\sum_{s=1}^{k-1}|\gamma_{skl}-\tilde\gamma_{skl}|\Big),\quad l\geqslant 1,$$

where $\gamma_{skl}:=F_{ks}(\rho_{l,k-1})$, $\tilde\gamma_{skl}:=\tilde F_{ks}(\tilde\rho_{l,k-1})$, $s=\overline{1,k-1}$, $k=\overline{2,n}$, $l\geqslant 1$. It is possible to choose $\tilde Q(x)$ such that $\xi_l=O(l^{-N-1})$ for $l\to\infty$.

For $\rho_0\in\Lambda^1$ we introduce the matrix $\tilde D(x,\rho_0,\rho)=[\tilde D_{sm}(x,\rho_0,\rho)]_{s,m=\overline{1,n}}$ by the formula

$$\tilde D(x,\rho_0,\rho):=\Big(\frac{\tilde\Phi^*(x,\mu)Q_0\tilde\Phi(x,\rho)}{\rho-\mu}\Big)^{\langle 0\rangle}_{|\mu=\rho_0}=\Big(\frac{\tilde\Phi^*_{\langle -1\rangle}(x,\rho_0)}{(\rho-\rho_0)^2}+\frac{\tilde\Phi^*_{\langle 0\rangle}(x,\rho_0)}{\rho-\rho_0}\Big)Q_0\tilde\Phi(x,\rho).$$

Denote $\rho_{lk0}=\rho_{lk}$, $\rho_{lk1}=\tilde\rho_{lk}$, $\gamma^{sk}_{l0}=\gamma_{skl}$, $\gamma^{sk}_{l1}=\tilde\gamma_{skl}$. We introduce the matrices

$$\varphi_{l\varepsilon}(x)=[\varphi^{ik}_{l\varepsilon}(x)]_{i=\overline{1,n},\,k=\overline{2,n}},\quad \tilde\varphi_{l\varepsilon}(x)=[\tilde\varphi^{ik}_{l\varepsilon}(x)]_{i=\overline{1,n},\,k=\overline{2,n}},$$

$$P_{l\varepsilon}(x,\rho)=[P^{km}_{l\varepsilon}(x,\rho)]_{k=\overline{2,n},\,m=\overline{1,n}},\quad \tilde P_{l\varepsilon}(x,\rho)=[\tilde P^{km}_{l\varepsilon}(x,\rho)]_{k=\overline{2,n},\quad m=\overline{1,n}},$$

$$G_{(l\varepsilon)(l_0\varepsilon_0)}(x)=[G^{km}_{(l\varepsilon)(l_0\varepsilon_0)}(x)]_{k,m=\overline{2,n}},\quad \tilde G_{(l\varepsilon)(l_0\varepsilon_0)}(x)=[\tilde G^{km}_{(l\varepsilon)(l_0\varepsilon_0)}(x)]_{k,m=\overline{2,n}},$$

$$l,l_0>1,\quad \varepsilon,\varepsilon_0=0,1,$$

by the formulae

$$\varphi_{l\varepsilon}^{ik}(x)=\Phi_{ik,\langle 0\rangle}(x,\rho_{l,k-1,\varepsilon}),\quad \tilde\varphi_{l\varepsilon}^{ik}(x)=\tilde\Phi_{ik,\langle 0\rangle}(x,\rho_{l,k-1,\varepsilon}),$$
$$P_{l\varepsilon}^{km}(x,\rho)=\sum_{s=1}^{k-1}\gamma_{l\varepsilon}^{sk}D_{sm}(x,\rho_{l,k-1,\varepsilon},\rho),\ \tilde P_{l\varepsilon}^{km}(x,\rho)=\sum_{s=1}^{k-1}\gamma_{l\varepsilon}^{sk}\tilde D_{sm}(x,\rho_{l,k-1,\varepsilon},\rho),$$
$$G_{(l\varepsilon)(l_0\varepsilon_0)}^{km}(x)=P_{l\varepsilon,\langle 0\rangle}^{km}(x,\rho_{l_0,m-1,\varepsilon_0}),\ \tilde G_{(l\varepsilon)(l_0\varepsilon_0)}^{km}(x)=\tilde P_{l\varepsilon,\langle 0\rangle}^{km}(x,\rho_{l_0,m-1,\varepsilon_0}).$$

Denote $u_{lm(x)}:=|\exp(\tilde\rho_{l,m-1}R_m x)|,\ l\in\tilde\Lambda_{m-1,j},\ j\in J_{m-1},\ R_m=R_m(S_j)$. Using (8) and taking our notations into account we obtain

$$\tilde\Phi(x,\rho)=\Phi(x,\rho)+\sum_{l\geqslant 1}\Big(\varphi_{l0}(x)\tilde P_{l0}(x,\rho)-\varphi_{l1}(x)\tilde P_{l1}(x,\rho)\Big). \tag{10}$$

It follows from (10) that

$$\tilde\varphi_{l_0\varepsilon_0}(x)=\varphi_{l_0\varepsilon_0}(x)+\sum_{l\geqslant 1}\Big(\varphi_{l0}(x)\tilde G_{(l0)(l_0\varepsilon_0)}(x)-\varphi_{l1}(x)\tilde G_{(l1)(l_0\varepsilon_0)}(x)\Big), \tag{11}$$
$$l_0>1,\quad \varepsilon_0=0,1.$$

For each fixed $x\in[0,T]$, relation (11) can be considered as a system of linear equations with respect to $\varphi_{l\varepsilon}(x)$, $l\geqslant 1$, $\varepsilon=0,1$. But the series in (11) converges only “with brackets”. Therefore, it is not convenient to use (11) as a main equation of the inverse problem. Below we will transfer (11) to a linear equation in a corresponding Banach space of sequences (see (15)).

For this purpose we introduce the matrices

$$\psi_{l\varepsilon}(x)=[\psi_{l\varepsilon}^{ik}(x)]_{i=\overline{1,n},\,k=\overline{2,n}},\quad H_{(l\varepsilon)(l_0\varepsilon_0)}(x)=[H_{(l\varepsilon)(l_0\varepsilon_0)}^{km}(x)]_{k,m=\overline{2,n}},\quad l,l_0>1,\quad \varepsilon,\varepsilon_0=0,1,$$

by the formulae

$$\psi_{l0}^{ik}(x)=(\xi_l u_{lk}(x))^{-1}(\varphi_{l0}^{ik}(x)-\varphi_{l1}^{ik}(x)),\ \psi_{l1}^{ik}(x)=(u_{lk}(x))^{-1}\varphi_{l1}^{ik}(x), \tag{12}$$
$$H_{(l0)(l_00)}^{km}(x)=\xi_l u_{lk}(x)(\xi_{l_0}u_{l_0m}(x))^{-1}\Big(G_{(l0)(l_00)}^{km}(x)-G_{(l0)(l_01)}^{km}(x)\Big),$$
$$H_{(l0)(l_01)}^{km}(x)=\xi_l u_{lk}(x)(u_{l_0m}(x))^{-1}\,G_{(l0)(l_01)}^{km}(x),$$
$$H_{(l1)(l_00)}^{km}(x)=u_{lk}(x)(\xi_{l_0}u_{l_0m}(x))^{-1}\Big(G_{(l0)(l_00)}^{km}(x)-G_{(l0)(l_01)}^{km}(x)-G_{(l1)(l_00)}^{km}(x)+G_{(l1)(l_01)}^{km}(x)\Big),$$
$$H_{(l1)(l_01)}^{km}(x)=u_{lk}(x)(u_{l_0m}(x))^{-1}\Big(G_{(l0)(l_01)}^{km}(x)-G_{(l1)(l_01)}^{km}(x)\Big).$$

Similarly, we define the matrices $\tilde\psi_{l\varepsilon}(x)$ and $\tilde H_{(l\varepsilon)(l_0\varepsilon_0)}(x)$. Then

$$|\psi_{l\varepsilon}^{ik}(x)|\leqslant C,\quad |\tilde\psi_{l\varepsilon}^{ik}(x)|\leqslant C, \tag{13}$$
$$|H_{(l\varepsilon)(l_0\varepsilon_0)}^{km}(x)|,\ |\tilde H_{(l\varepsilon)(l_0\varepsilon_0)}^{km}(x)|\leqslant\frac{C\xi_l}{|l-l_0|+1}\,. \tag{14}$$

Let $V$ be a set of indices $u=(l,\varepsilon)$, $l\geqslant 1$, $\varepsilon=0,1$. For each fixed $x\in[0,T]$, we define the vector

$$\psi(x)=[\psi_u(x)]_{u\in V}=[\psi_{l0}(x),\psi_{l1}(x)]_{l\geqslant 1}=[\psi_{10},\psi_{11},\psi_{20},\psi_{21},\dots],$$

and the block matrix

$$H(x)=[H_{uv}(x)]_{u,v\in V}=\begin{bmatrix}H_{(l0)(l_00)}(x) & H_{(l0)(l_01)}(x)\\ H_{(l1)(l_00)}(x) & H_{(l1)(l_01)}(x)\end{bmatrix}_{l,l_0\geqslant 1},$$

$$u = (l, \varepsilon), \quad v = (l_0, \varepsilon_0), \quad \varepsilon, \varepsilon_0 = 0, 1.$$

Analogously, we define the matrices $\tilde{\psi}(x)$ and $\tilde{H}(x)$. In view of our notations, relation (11) transforms to the form

$$\tilde{\psi}_{l_0\varepsilon_0}(x) = \psi_{l_0\varepsilon_0}(x) + \sum_{l,\varepsilon} \psi_{l\varepsilon}(x)\tilde{H}_{(l\varepsilon)(l_0\varepsilon_0)}(x), \quad l, l_0 \geqslant 1, \quad \varepsilon, \varepsilon_0 = 0, 1,$$

or, which is the same,

$$\tilde{\psi}(x) = \psi(x)(I + \tilde{H}(x)). \tag{15}$$

According to (13) and (14), the series in (15) converges absolutely and uniformly in $x \in [0, T]$.

Starting from (8), we arrived at (15). By similar arguments, starting from (9) one can get the relation

$$(I + \tilde{H}(x))(I - H(x)) = I.$$

Interchanging places for $Q$ and $\tilde{Q}$, we obtain

$$(I - H(x))(I + \tilde{H}(x)) = I.$$

Let us consider the Banach space $m$ of bounded sequences $\alpha = [\alpha_u]_{u \in V}$ with the norm $\|\alpha\| = \sup_{u \in V} |\alpha_u|$. It follows from (14) that for each fixed $x \in [0, T]$, the operators $I + \tilde{H}(x)$ and $I - H(x)$, acting from $m$ to $m$, are linear bounded operators, and

$$\|H(x)\|, \|\tilde{H}(x)\| \leqslant C \sup_{l \geqslant 1} \sum_{l_0 \geqslant 1} \frac{\xi_l}{|l - l_0| + 1} < \infty.$$

Thus, we have proved the following theorem

**Theorem 2.** *For each fixed $x \in [0, T]$, the vector $\psi(x) \in m$ satisfies the equation (15) in the Banach space $m$. Moreover, the operator $I + \tilde{H}(x)$ has a bounded inverse operator, i.e. equation (15) is uniquely solvable.*

Equation (15) is called the *main equation* of the inverse problem. Solving (15) we find the vector $\psi(x)$, and consequently, the functions $\varphi_{l\varepsilon}(x)$. Then, by (10) we calculate $\Phi(x, \rho)$. Denote

$$\varepsilon(x) = \frac{1}{2\pi i} \int_\omega \left( \Phi(x, \mu)\tilde{\Phi}^*(x, \mu)Q_0 - Q_0\Phi(x, \mu)\tilde{\Phi}^*(x, \mu) \right). \tag{16}$$

**Theorem 3.** *The following relation holds*

$$Q(x) = \tilde{Q}(x) + \varepsilon(x). \tag{17}$$

**Proof.** Differentiating (8) and using (7), we calculate

$$\tilde{\Phi}'(x, \rho) = \Phi'(x, \rho) + \frac{1}{2\pi i} \int_\omega \Phi'(x, \mu)\tilde{\Phi}^*(x, \mu)Q_0\tilde{\Phi}(x, \rho) \frac{d\mu}{\mu - \rho} - \\ - \frac{1}{2\pi i} \int_\omega \Phi(x, \mu)\tilde{\Phi}^*(x, \mu)\tilde{\Phi}(x, \rho) d\mu.$$

Since $\Phi(x, \rho)$ is a solution of system (1) it follows that $\Phi'(x, \rho) = (\rho B_0 - B(x))\Phi(x, \rho)$, where $B_0 = Q_0^{-1}$, $B(x) = B_0 Q(x)$. Similarly, $\tilde{\Phi}'(x, \rho) = (\rho B_0 - \tilde{B}(x))\tilde{\Phi}(x, \rho)$. This yields

$$(\rho B_0 - \tilde{B}(x))\tilde{\Phi}(x, \rho) = (\rho B_0 - B(x))\Phi(x, \rho) + \\ + \frac{1}{2\pi i} \int_\omega (\mu B_0 - B(x))\Phi(x, \mu)\tilde{\Phi}^*(x, \mu)Q_0\tilde{\Phi}(x, \rho) \frac{d\mu}{\mu - \rho} - \frac{1}{2\pi i} \int_\omega \Phi(x, \mu)\tilde{\Phi}^*(x, \mu)\tilde{\Phi}(x, \rho) d\mu.$$

Replacing here $\Phi(x,\rho)$ from (8), we calculate

$$\hat{B}(x)\tilde{\Phi}(x,\rho)=\frac{1}{2\pi i}\int_\omega B_0\Phi(x,\mu)\tilde{\Phi}^*(x,\mu)Q_0\tilde{\Phi}(x,\rho)d\mu-\frac{1}{2\pi i}\int_\omega \Phi(x,\mu)\tilde{\Phi}^*(x,\mu)\tilde{\Phi}(x,\rho)d\mu.$$

Multiplying this relation from the left on $Q_0$, we obtain $\hat{Q}(x)\tilde{\Phi}(x,\rho)=\varepsilon(x)\tilde{\Phi}(x,\rho)$, and consequently, (17) holds. □

The solution of the inverse problem can be found by the following algorithm.

**Algorithm.** Given the Weyl matrix $M(\rho)$.

1. Choose $\tilde{Q}(x)$, and construct $\tilde{\psi}(x)$ and $\tilde{H}(x)$.
2. Find $\psi(x)$ by solving equation (15).
3. Calculate $\Phi(x,\rho)$ via (10), where $\varphi_{l\varepsilon}(x)$ is constructed from (12).
4. Construct $Q(x)$ by (17).

Let us now formulate the necessary and sufficient conditions for the solvability of the inverse problem. Denote by $\mathbf{W}$ the set of meromorphic matrices $M(\rho)=[M_{mk}(\rho)]_{m,k=\overline{1,n}}$, $M_{mk}(\rho)\equiv\delta_{mk}$ for $m\geqslant k$, having only simple poles $\Lambda=\bigcup_m \Lambda_m$ (in general, the set $\Lambda$ is different for each matrix $M(\rho)$) and such that (5) is valid and for each $\rho_0\in\Lambda$ the properties $S_1$ and $S_2$ hold.

Clearly, if $\ell\in V_N$ and $M(\rho)$ is the Weyl matrix for $\ell$, then $M(\rho)\in\mathbf{W}$.

**Theorem 4.** *A matrix $M(\rho)\in\mathbf{W}$ is the Weyl matrix for some $\ell\in V_N$ if and only if the following conditions hold:*

*1) (asymptotics) there exists $\tilde{\ell}\in V_N$ such that $\sum_l \xi_l<\infty$;*

*2) (condition S) for each fixed $x\in[0,T]$ the linear bounded operator $I+\tilde{H}(x)$, acting from m to m, has a bounded inverse operator;*

*3) $\varepsilon(x)\in W_N$, where $\varepsilon(x)$ is defined by* (16).

*Under these conditions, the potential $Q(x)$ is constructed by Algorithm.*

The necessity part is proved above. In the sufficiency part, we have a matrix $M(\rho)\in\mathbf{W}$ satisfying the conditions of Theorem 4. Using the Algorithm we construct the potential $Q(x)$, i.e. we construct $\ell\in V_N$. By similar arguments as in [8], one can check that the matrix $M(\rho)$ is the Weyl matrix for $\ell$.

**Remark.** The inverse problem from a system of spectra.

The zeros $\Lambda_{mk}:=\{\rho_{lmk}\}$ of the entire functions $\Delta_{mk}(\rho)$ coincide with the eigenvalues of the boundary value problems $L_{mk}$ for system (1) with the boundary conditions

$$U_1(Y)=\ldots=U_{m-1}(Y)=U_k(Y)=V_1(Y)=\ldots=V_{n-m}(Y)=0.$$

The inverse problem of recovering the potential from the system of spectra is formulated as follows: given the spectra $\Lambda_m$ and $\Lambda_{mk}$ of the boundary value problems $L_m$ and $L_{mk}$ ($m=\overline{1,n-1}$, $k>m$), construct the potential $Q(x)$. Since the functions $\Delta_{mk}(\rho)$ are uniquely determined by their zeros, it follows from (2) that this inverse problem can be reduced to the inverse problem from the Weyl matrix.

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