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Article

Numerical solution of first-order exact differential equations by the integrating factor method

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Abstract. A numerical algorithm for solving exact differential equations is proposed, based both on the efficient calculation of integrating factors and on a "new" numerical method for integrating functions. Robust determination of the integrating factors is implemented by using the Chebyshev interpolation of the desired functions and performing calculations on Gauss – Lobatto grids, which ensure the discrete orthogonality of the Chebyshev matrices. After that, the integration procedure is carried out using the Chebyshev integration matrices. The integrating factor and the final potential of the ODE solution are presented as interpolation polynomials depending on a limited number of numerically recoverable expansion coefficients.

Keywords: spectral method, collocation, integrating factors, integration matrices, recovery of coefficients, inverse problem

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Численное решение дифференциальных уравнений первого порядка в полных дифференциалах методом интегрирующего множителя

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Аннотация. Предложен численный алгоритм решения дифференциальных уравнений в полных дифференциалах, основанный как на эффективном вычислении интегрирующих множителей, так и на «новом» численном методе интегрирования функций. Устойчивое определение интегрирующих множителей обеспечивается за счет использования чебышевской интерполяции искомых функций и проведения расчетов на сетках Гаусса – Лобатто, обеспечивающих дискретную ортогональность чебышевских матриц. После чего процедура интегрирования осуществляется с помощью чебышевских матриц интегрирования. Интегрирующий множитель и итоговый потенциал решения обыкновенного дифференциального уравнения представляются в виде интерполяционных полиномов, зависящих от ограниченного количества численно восстанавливаемых коэффициентов разложения.

Ключевые слова: спектральный метод, коллокация, интегрирующие множители, матрицы интегрирования, восстановление коэффициентов, обратная задача

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Introduction

On the way of generalizing the pseudospectral method of Chebyshev collocations for solving linear ODEs to the class of nonlinear ODEs, we dwell on an intermediate subclass of special linear differential equations with two independent variables — exact differential equations [1].

The problem of numerical integration of ODEs is traditionally studied within the framework of solving the Cauchy problem. The most common Runge – Kutta method is related to all other methods included in the conventional libraries. Finite-difference methods for solving differential equations and finite element methods reduce the original continuous problem to a discrete analog, a system of algebraic equations. For linear differential equations (DE), this is a system of linear algebraic equations (SLAE). Both approaches in their traditional implementations require significant amounts of memory and/or substantial time costs.

An alternative to the latter approaches, which can be considered finite-dimensional approximations of the desired solutions and coefficients of equations from infinite-dimensional function spaces, is a truncated expansion in complete systems of orthogonal polynomials. For them, three-term relations are performed, which make it possible to drastically reduce the number of intermediate calculations. Another resource is the availability of special “orthogonal” discrete grids consisting of the roots of the corresponding polynomials. An example is the Gauss – Lobatto grid for Chebyshev polynomials. Ultimately, these factors reduce the required memory and computation time by orders of magnitude.

The method of integrating factors is a representation of the solution in the form of a product of functions, a kind of method of separation of variables. Representing a solution as a product of two or more functions, compared to methods for finding a single desired function, provides much more opportunities and more variety to obtain the desired solution. The approach to the numerical solution of ODEs based on integrating factors offers new possibilities compared to simple traditional methods.

In analytical form, the method of integrating factors for constructing solutions of both linear and nonlinear differential equations has been used for a long time [2]. We propose an efficient numerical approach based on the collocation method and the use of integrating factors. A robust algorithm for the search for solution is implemented in the spectral space, and for the polynomial interpolation of the solution, expansion into a series of Chebyshev polynomials of the first kind is used.

1. Motivation

The class of exact differential equations or total differential equations is a kind of ordinary differential equations that are widely used in physics and technology (see, e.g., papers by L. L. Doskolovich [3–6] about inverse problems of calculating optical elements). In Refs. [7, 8], the inverse problem of designing a reflective surface of arbitrary shape to create a given illumination distribution is reduced to the form of the Monge – Ampère problem. Prior to that, in the papers by Doskolovich et al. [9, 10], this problem was solved in the presence of the surface axial symmetry and for one-dimensional illumination distributions.

In Ref. [11], a new method for reconstructing a reflecting (refracting) surface from a given source–target map is proposed, which determines the relationship between the directions of incident and reflected (refracted) rays. In the proposed method, the optical surface is represented as an envelope of multiple paraboloids (reflecting surface) or ellipsoids (refractive surface). This representation makes it possible to reduce the problem of designing an optical surface to restoring a function from its total differential. The proposed approach is illustrated by the synthesis of mirrors that produce uniform illumination on a square target in the far-field zone. The results of the calculations showed that the proposed method allows forming qualitative illumination distributions even if the integrability condition is not met.

Reference [12] proposes a method for designing reflective surfaces that form given continuous illumination distributions in two-dimensional regions. The surface of the mirror is represented as an envelope of a two-parameter family of ellipsoids. The first focus of each ellipsoid coincides with the point light source, and the second is in the illuminated area. This surface representation can be interpreted as an extreme case of a segmented surface used in the support quadric method to focus on a set of points. The envelope equation depends on the function that determines the lengths of the major axes of the ellipsoids of the family. The calculation of this function is carried out using a continuous approximation of the discrete function obtained from the solution of the discrete problem of focusing on a set of points. The high efficiency of the proposed method is illustrated by the developed examples of mirrors for creating a uniform distribution of illumination in areas of various shapes.

Based on the results of Refs. [11, 12], we developed a method for restoring a reflective (refractive) surface from a given source-screen mapping. This representation makes it possible to reduce the problem of designing an optical surface to the problem of restoring a function from its total differential. In this case, reduction to a total differential can be carried out using an integrating factor.

2. Basic solution method

As the main construction in developing a method for solving the nonlinear ODEs considered below, we will use a stable and efficient algorithm for restoring a function from a known derivative (practically this is the equivalent of the problem of restoring the antiderivative from the integrand) — the solution of the Cauchy problem for an ordinary differential equation [13, 14]

$$y'(x) = f(x), \quad y(x_0) = y_0, \quad x \in [a, b].$$

The method consists in representing the approximate solution of the problem in the form of a series $y(x) = \sum_{k=0}^{n} c_k T_k(x)$ of Chebyshev polynomials of the first kind $T_k(x)$ with the domain of definition $x \in [-1, 1]$. The linear transformation $l(x) = \frac{(2x-(b+a))}{(b-a)}$ allows to proceed to solving the problem in the interval $[-1, 1]$, the upper estimate of the interpolation error [15] having the form:

$$|y(x) - \sum_{k=0}^{n} c_k T_k(x)| <= \frac{1}{2^n (n+1)!} |\frac{(b-a)}{2}|^{n+1} \max_{\xi \in [a,b]} |y^{(n+1)}(\xi)|.$$

Thus, the Chebyshev interpolation is an almost optimal approximation in the sense of the norm L_∞ nearly equivalent to the L_2 norm. Moreover, the use of Gauss – Lobatto nodes as interpolation nodes leads to optimal integration formulas.

Below we consider both the interpolation problem and the problem of solving ODE exactly in the interval $[-1, 1]$.

$$y'(x) = f(x), \quad y(x_0) = y_0, \quad x \in [-1, 1]. \tag{1}$$

We propose to find the expansion coefficients $c_k, k = 0, \ldots, n$ of the approximate solution (1) in the form of the series $y(x) = \sum_{k=0}^{n} c_k T_k(x)$ of Chebyshev polynomials in two stages.

At the first stage, the stage of derivative interpolation, the collocation method is used to calculate the coefficients b_k, $k = 0, \ldots, n$ of the derivative expansion $f(x_j) = \sum_{k=0}^{n} b_k T_k(x_j)$, $j = 0, \ldots, n$ in the orthogonal basis of the Chebyshev polynomials of the first kind.

The algorithm stability is achieved at the expense of the discrete orthogonality of the modified Chebyshev matrix $\mathbf{T} = [T_{j,k}]_{0 \leqslant j,k \leqslant n}$ on the Gauss – Lobatto grid. The choice of collocation points $x_j = \cos(\pi j/n)$, $j = 0, \ldots, n$, makes it possible by multiplying the first and the last equation of the collocation method by $1/\sqrt{2}$ to obtain an equivalent "modified" system with a new matrix $\tilde{\mathbf{T}}$ instead of $\mathbf{T}$ and vector $\tilde{\mathbf{f}}$ instead of $\mathbf{f}$. The new system already possesses the property of discrete "orthogonality". Its multiplication from the left by the transposed matrix $\tilde{\mathbf{T}}^{\mathbf{T}}$ yields a system with the diagonal matrix

$$\begin{bmatrix} n & 0 & 0 & \ldots & 0 \\ 0 & \frac{n}{2} & 0 & \ldots & 0 \\ 0 & 0 & \frac{n}{2} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ddots & \ldots \\ 0 & 0 & 0 & \ldots & n \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \ldots \\ b_n \end{bmatrix} = \tilde{\mathbf{T}}^{\mathbf{T}} \begin{bmatrix} \tilde{f}_0 \\ \tilde{f}_1 \\ \tilde{f}_2 \\ \ldots \\ \tilde{f}_n \end{bmatrix}$$

where $\tilde{\mathbf{f}} = \tilde{\mathbf{T}}^{\mathbf{T}} \left(f_0/\sqrt{2}, f_1, \ldots, f_{n-1}, f_n/\sqrt{2} \right)^{\mathbf{T}}$

The expansion coefficients for the function $f(x)$ are easily expressed in the explicit form

$$b_0 = \frac{\tilde{f}_0}{n}, \quad b_1 = \frac{2\tilde{f}_1}{n}, \quad b_2 = \frac{2\tilde{f}_2}{n}, \quad \ldots, \quad b_n = \frac{\tilde{f}_n}{n}.$$

The second stage implies the calculation of the antiderivative expansion coefficients c_k, $k = 1, \ldots, n$. For this purpose, we multiply the banded three-diagonal integration matrix [13, 16, 17] by the vector of the derivative expansion coefficients b_k, $k = 0, \ldots, n$ and obtain the expansion coefficients c_k, $k = 1, \dots, n$ of the antiderivative, except the zeroth one, c_0:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & \frac{-1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & \frac{-1}{4} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \ddots & \frac{1/2}{(n-1)} & 0 & \frac{-1/2}{(n-1)} \\ 0 & 0 & 0 & \ddots & 0 & \frac{1}{2n} & 0 \end{bmatrix} \times \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix} = \begin{bmatrix} c_0 = ? \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ \vdots \\ c_{n-1} \\ c_n \end{bmatrix}.$$

In some cases, e.g., when solving the Cauchy problem with a given initial condition y_0, it is also necessary to know the zero coefficient c_0. Then a linear equation that specifies the initial/boundary condition is additionally solved, and the coefficient c_0 is calculated by the formula $c_0 = y_0 - \sum_{k=1}^{n} c_k(-1)^k$.

The approach presented has been successfully applied by the authors to solve linear ODEs of the first and second order [13, 14].

The third stage is the ultimate calculation of the potential surface values, which requires efficient computation of definite integrals of the functions, interpolation coefficients of which are already known. We will present a formula for calculating definite integrals from the known coefficients of Chebyshev interpolation, which directly follows from the recurrence relation for the Chebyshev polynomials of the first kind.

Assertion. Let the coefficients $b_k,\ k = 0, \dots, n$ be the coefficient of expansion of the integrand $f(x) = \sum_{k=0}^{n} b_k T_k(x),\ j = 0, \dots, n$ in Chebyshev polynomials. Then the exact formula for calculating a definite integral by the interpolation coefficient of function $f(x)$ expansion in the interval $[-1, 1]$ has the form:

$$\int_{-1}^{1} f(x)dx \approx \int_{-1}^{1} \sum_{k=0}^{n} c_k T_k(x)dx = 2 \sum_{k=0,\ k=even}^{n} \frac{c_k}{1-k^2}.$$

On the way to generalizing the pseudospectral method of Chebyshev collocations from linear ODEs to the class of nonlinear ODEs, we dwell on an intermediate subclass of special linear differential equations with two independent variables.

3. Equations with separable variables

Consider the solution of one of the simplest nonlinear equations of the first order — an equation with separable variables.

Nonlinear first-order general equation

$$\frac{dy}{dx} = f(x, y)$$

can be written in the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0.$$

A transformation to such form is always possible. In the particular case when $M(x, y)$ depends only on x, and $N(x, y)$ depends only on y, the equation is reduced to the form

$$M(x) + N(y)\frac{dy}{dx} = 0 \tag{2}$$

of an equation with separable variables. It is possible to use the symmetry in the form of the equation and emphasize the 'independence' of the dependent variables from the independent ones:

$$M(x)dx + N(y)dy = 0. \tag{3}$$

The solution (a general integral) of this equation with separable variables in an implicit form is obtained by integrating Eq. (3):

$$\int M(x)dx + \int N(y)dy = C, \tag{4}$$

where C is an arbitrary constant. Any differentiable function $y = \varphi(x)$, satisfying condition (4) is a solution of Eq. (2). Thus, it implicitly defines the solution of the differential equation with separable variables.

This form of the solution of the separable equation (with separable variables) can be substantiated in the following way. We denote by H_1 and H_2 the functions, the derivatives of which are M and N respectively, i.e.,

$$H_1'(x) = M(x), \quad H_2'(y) = N(y),$$

then Eq. (2) can be rewritten as

$$H_1'(x) + H_2'(y)\frac{dy}{dx} = 0. \tag{5}$$

According to the rule of differentiating a composite function, a chain of equalities is valid

$$H_2'(y)\frac{dy}{dx} = \frac{d}{dy}H_2(y)\frac{dy}{dx} = \frac{d}{dx}H_2(y)$$

and, therefore, Eq. (5) can be presented as

$$\frac{d}{dx}\left[H_1(x) + H_2(y)\right] = 0. \tag{6}$$

Integrating the latter, we get the general solution

$$H_1(x) + H_2(y) = c, \tag{7}$$

where c is an arbitrary constant. Any differentiable function $y = \varphi(x)$, satisfying condition (6) is a solution to Eq. (2) with separable variables in the implicit form.

Differential equation (2) together with the initial condition

$$y(x_0) = y_0 \tag{8}$$

defines a Cauchy problem. The solution of such an initial-value problem implies the specification of a certain numerical value of the parameter c in Eq. (7). It is possible to specify such a value by substituting in Eq. (7) the values $x = x_0$ and $y = y_0$ and computing the desired value of the constant

$$c = H_1(x_0) + H_2(y_0).$$

Substituting the calculated value of the constant c in (7) and keeping in mind that

$$H_1(x) - H_1(x_0) = \int_{x_0}^{x} M(s)\,ds, \quad H_2(y) - H_2(y_0) = \int_{y_0}^{y} N(s)\,ds,$$

we obtain a formula to determine the particular integral curve of Eq. (2), passing through the given point.

The solution of the Cauchy problem satisfying the initial condition $y(x_0) = y_0$ is determined by the relation

$$\int_{x_0}^{x} M(\xi)d\xi + \int_{y_0}^{y} N(\eta)d\eta = 0, \tag{9}$$

that determines the integral curve passing through the point (x_0, y_0).

Equation (9) implicitly determines the solution of differential equation (2), satisfying initial condition (8). It should be kept in mind that in order to find an explicit formula that describes the behavior of the integral curve, it is necessary to express the dependent variable y as a function of x from the nonlinear equation (9). Unfortunately, it is often impossible to do analytically; in such cases it is possible to use numerical methods to find approximate values of $y(x)$.

The algorithm of definite integral calculation based on Chebyshev interpolation:

- by linear transformation $l(x) = \frac{2x-(b+a)}{b-a}$ reduce the integral calculation to the interval $[-1, 1]$;
- calculate the coefficients b_k, $k = 0, \dots, n$ of the integrand expansion $f(x) = \sum_{k=0}^{n} b_k T_k(x)$, $j = 0, \dots, n$ in Chebyshev polynomials;
- calculate the values of the potential function at the point (x, y) under the given initial condition $y(x_0) = y_0$:

$$F(x,y) = \int_{x_0}^{x} M\, d\xi + \int_{y_0}^{y} N(\eta)\, d\eta$$

  using the exact formula for calculating a definite integral from the interpolation coefficients of the expansion of the integrand $f(x)$:

$$\int_{-1}^{1} f(x)\, dx \approx \int_{-1}^{1} \sum_{k=0}^{n} c_k T_k(x)\, dx = 2 \sum_{k=0,\ k=even}^{n} \frac{c_k}{1-k^2}.$$

4. Numerical solution of an exact differential equation

Consider a simply connected open subset D of R^2 and two functions M and N, continuous in D. *An implicit ordinary differential equation of the first order*

$$M(x,y)\, dx + N(x,y)\, dy = 0 \tag{10}$$

is called *an exact differential equation (a total differential equation)*, if there exists a continuously differentiable function $F(x,y)$, *called a potential function*, such that

$$\frac{dF(x,y)}{dx} = M(x,y), \quad \frac{dF(x,y)}{dy} = N(x,y).$$

The integration of such an equation reduces to constructing the function $F(x,y)$, after which the solution is found in the form $F(x,y) = C$, since $dF = 0$. Thus, to solve the problem it is necessary to calculate the values of the integral curve, which is a line of intersection of the potential surface with a horizontal plane.

Let the function $F(x,y)$ be a total differential of an exact DE defined on a certain simply connected and open subset D of R^2. Then the differentiable function $f(x)$ such that $(x, f(x)) \in D$ is a solution if and only if there exists a real number c, such that

$$F(x, f(x)) = c.$$

In the case when the solution is subject to the requirement of passing through a given point (a problem with an initial or boundary condition),

$$y(x_0) = y_0,$$

the local value of the potential function can be calculated by the formula [18, Lesson 23]:

$$F(x,y) = \int_{x_0}^{x} M(t,y_0)\, dt + \int_{y_0}^{y} N(x,t)\, dt =$$
$$= \int_{x_0}^{x} M(t,y_0)\, dt + \int_{y_0}^{y} \left[N(x_0,t) + \int_{x_0}^{x} \frac{\partial M}{\partial t}(u,t)\, du \right] dt. \tag{11}$$

Solving the implicit equation $F(x,y) = c$ with respect to y, where c is a given constant, allows us to calculate all possible solutions (Fig. 1).

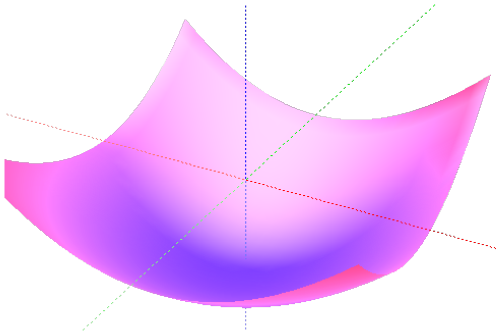


Fig. 1. The potential $F(x, y) = \left(x^2 + y^2\right)/2$ of the exact DE $x\,dx + y\,dy = 0$ restored by the formula (11) (color online)

5. Integrating factors

The method of integrating factors allows a generalization of the proposed construction to a wider class of nonlinear first-order ODEs reducible to the total differential form.

Three cases of equations admitting exact solutions are known:

- based on trivial integrating factors: $\mu \equiv 1$;
- with an integrating factor depending only on x: $\mu(x)$;
- with an integrating factor depending only on y: $\mu(y)$.

In all three cases considered, the search for particular solutions is based on the pseudospectral Chebyshev collocation method. The integrating factor method for implementing the algorithm of numerical solution of linear ODEs by means of integration matrices [13] allows reducing the problem to twofold sequential multiplication of a two-diagonal integration matrix by the vector of spectral coefficients of the derivative. The lacking constants of integration are determined from either boundary, or initial conditions of the problem. In the second and third case, a nontrivial problem of finding the integration factors numerically additionally arises.

Case 1

If the left–hand side of the differential equation (10) is a total differential, i.e.,

$$M_y(x, y) - N_x(x, y) = 0,$$

then both its terms are calculated separately by means of the appropriate integration matrices.

If the differential equation (10) is not exact, but still has a solution, then a function $\mu(x, y)$ necessarily exists, such that the equivalent equation obtained by multiplying both sides of Eq. (10) by $\mu(x, y)$

$$(\mu M)\,dx + (\mu N)\,dy = 0$$

is exact. Such function $\mu(x, y)$ is called an integrating factor of the initial equation. Experience shows finding an integrating factor in the most general form is extremely difficult. Below we consider two special particular cases.

Case 2

If $M_y(x, y) - N_x(x, y) \neq 0$, and $\frac{(M_y - N_x)}{N}$ is a function only of x, let us denote it by $\xi(x)$. Then

$$\mu(x) = \pm \exp\left(\int \xi(x)\,dx\right)$$

is an integrating factor of this differential equation.

Case 3

If $M_y(x,y) - N_x(x,y) \neq 0$, and $\frac{(M_y - N_x)}{(-M)}$ is a function only of y, let us denote it by $\psi(y)$. Then

$$\mu(y) = \pm\exp\left(\int \psi(y)\, dy\right)$$

is an integrating factor of this differential equation.

A more general situation, when none of these cases takes place, is not considered here.

6. Discussion, numerical examples

Let us consider examples of the three cases listed in the previous section.

Example 1. The case of total differential

Consider a numerical method for constructing a general integral (potential) of an implicit exact differential equation:

$$(\sin(xy)xy\cos xy)\ dx + x^2\cos xy\, dy = 0. \tag{12}$$

Equation (12) is an exact equation if and only if the condition

$$\frac{dM(x,y)}{dy} \equiv \frac{dN(x,y)}{dx} \tag{13}$$

is valid in some simply connected domain R of variation of variables.

If allowed by the problem statement, we analytically check the fulfillment of the condition (13):

$$\frac{dM(x,y)}{dy} = \frac{d}{dy}(\sin(xy)xy\cos xy) = 2x\cos xy - x^2y\sin xy,$$
$$\frac{d(x,y)}{dx} = \frac{d}{dx}\left(x^2\cos xy\right) = 2x\cos xy - x^2y\sin xy.$$

For Eq. (12), the necessary and sufficient condition is satisfied, and it is an exact differential equation.

In the case when checking the condition (13) analytically is not possible, it may well be sufficient to check this condition numerically. The feasibility of the necessary and sufficient conditions can be checked on a fine enough grid in the domain R with respect to the variables $(x,y) \in R$ by numerical pointwise comparison of the computed derivatives. When approximating functions with series expansion in Chebyshev polynomials of the first kind, such a comparison can be efficiently carried out using Gauss – Lobatto grids in both variables and Chebyshev differentiation matrices in the spectral space.

Next we substitute specific expressions $M(x,y) = \sin(xy)xy\cos xy$ and $N(x,y) = x^2\cos xy$ into Eq. (11) to calculate the potential at the initial value, $y(0) = 0, x_0 = 0, y_0 = 0$. In the vicinity of the initial point, the potential values are calculated using the formula

$$F(x,y) = \int_{x_0}^{x} \left(\sin(ty_0)\, ty_0\cos ty_0\right)\, dt +$$
$$+ \int_{y_0}^{y}\left[{x_0}^2\cos x_0t + \int_{x_0}^{x}\left\{2u\cos ut - u^2t\sin ut\frac{\partial M}{\partial t}(u,t)\right\} du\right] dt.$$

Numerically, the integrals in this equation are calculated based on the method of restoring the antiderivative from the known integrand. Plots of the calculated potential and its errors are shown in Fig. 2.

The exact solution, the potential function of Eq. (12) looks as follows:

$$x\sin(xy) = c.$$

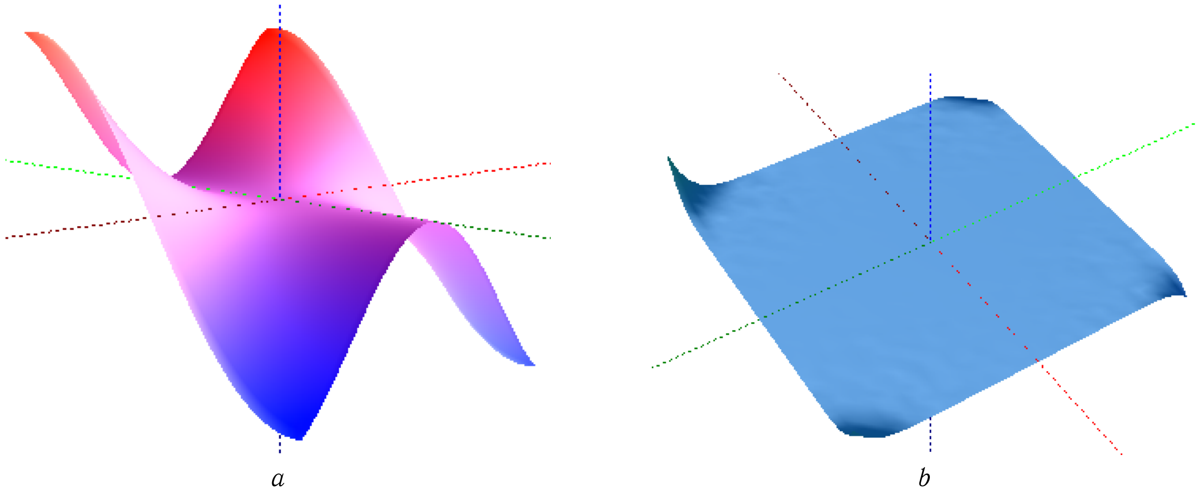


Fig. 2. The calculated potential surface and the error of its calculation: a — potential surface, the interval from -1.5 to $+1.5$ is shown; b — deviation from the exact surface $(F(x,y) - x\sin(xy)) \cdot 10^{12}$ (color online)

Example 2

The case of reduction to the form of a total differential using the integrating factor $\mu(x)$ [17, Sample 2.6.1].

Consider the equation

$$\left(2xy^3 - 2x^3y^3 - 4xy^2 + 2x\right)\,dx + (3x^2y^2 + 4y)\,dy = 0. \tag{14}$$

We will search the solution by choosing an appropriate integrating factor.

Passing to the standard notation in Eq. (14):

$$M = 2xy^3 - 2x^3y^3 - 4xy^2 + 2x,$$
$$N = 3x^2y^2 + 4y$$

we write the difference of derivatives in the form

$$M_y - N_x = 6xy^2 - 6x^3y^2 - 8xy - 6xy^2 = -6x^3y^2 - 8xy. \tag{15}$$

Since the right-hand side of Eq. (15) is nonzero, Eq. (14) is not exact. However, the expression

$$\frac{M_y - N_x}{N} = -\frac{6x^3y^2 + 8xy}{3x^2y^2 + 4y} = -2x$$

is independent of y, therefore, the integrating factor can be calculated by the formula

$$\mu(x) = \exp\left(-\int 2x\,dx\right)$$

and equals $\mu(x) = \exp\left(-x^2\right)$. The numerical determination of the integrating factor reduces, as above, to the method of calculating the antiderivative of the integrand $(M_y - N_x)/N$, followed by the calculation of the exponential function in necessary points of the desired range. Multiplying Eq. (14) by $\mu(x)$, we proceed to the solution of the equivalent exact equation

$$e^{-x^2}\left(2xy^3 - 2x^3y^3 - 4xy^2 + 2x\right)\,dx + e^{-x^2}\left(3x^2y^2 + 4y\right)\,dy = 0.$$

To solve this equation, it is necessary to construct a function (potential) $F(x,y)$, such that

$$F_x(x,y) = e^{-x^2}(2xy^3 - 2x^3y^3 - 4xy^2 + 2x), \tag{16}$$

$$F_y(x,y) = e^{-x^2}\left(3x^2y^2 + 4y\right). \tag{17}$$

Substituting into the formula (11) expressions (16), (17), and the mixed derivative

$$F_{xy}(x,y) = e^{-x^2}\left(6xy^2 - 6x^3y^2 - 8xy\right),$$

for each desired point $(x,y) \in R$ of the domain of solution existence at the given initial condition $y(0) = 0, x_0 = 0, y_0 = \sqrt{1/2}$, we build the approximating potential surface (18) (Fig. 3).

The exact solution has the form

$$F(x,y) = e^{-x^2}\left(y^2\left(x^2y + 2\right) - 1\right). \tag{18}$$

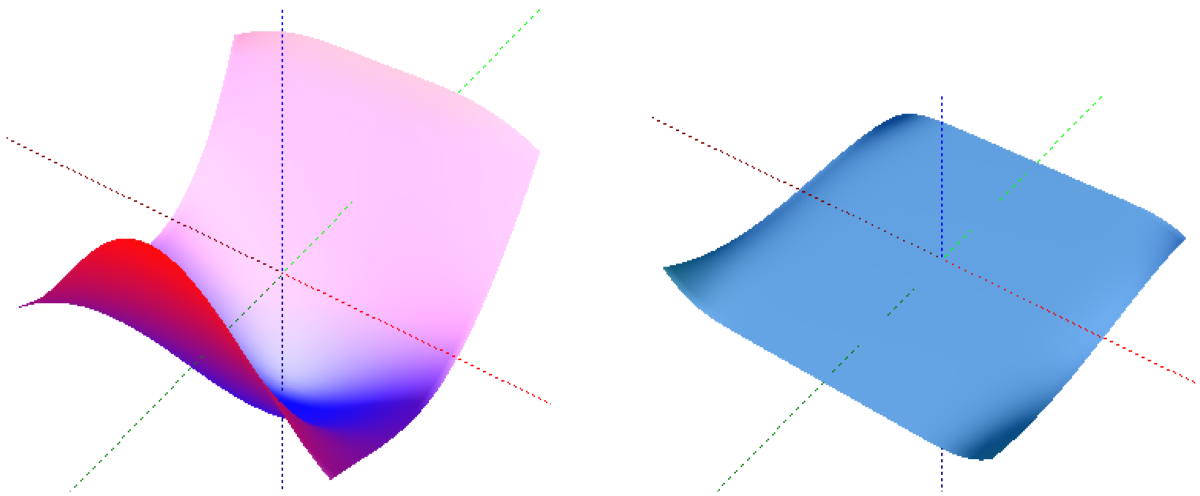


Fig. 3. The calculated potential surface and the error of its calculation: *a* — potential surface, the interval from -1.0 to $+1.5$ is shown; *b* — deviation from the exact solution $\Big(F(x,y) - e^{-x^2}\left(y^2\left(x^2y+2\right)-1\right)\Big) \cdot 10^8$ (color online)

Example 3

The case of reducing the initial equation to the total differential form using an integrating factor $\mu(y)$ [17, Sample 2.6.2]:

$$2xy^3\,dx + (3x^2y^2 + x^2y^3 + 1)\,dy = 0. \tag{19}$$

Let us introduce the standard notation in Eq. (19)

$$M = 2xy^3, \quad N = 3x^2y^2 + x^2y^3 + 1,$$

then the criterion of the equation belonging to exact differential equations has the form

$$M_y - N_x = 6x^2 - \left(6xy^2 + 2xy^3\right) = -2xy^3. \tag{20}$$

Since the right-hand side of Eq. (20) is nonzero, Eq. (19) is not exact. However, the expression

$$\frac{M_y - N_x}{N} = -\frac{2xy^3}{3x^2y^2 + x^2y^3 + 1}$$

depends on both x and y, therefore, it turns out impossible to calculate the integrating factor in the form $\mu(x)$.

Let us try another possibility of constructing the integrating factor — as a function of the second variable:

$$\frac{N_x - M_y}{M} = -\frac{2xy^3}{2xy^3} = 1.$$

Since the relation (13) is independent of x, the integrating factor $\mu(y)$ exists and is written as $\mu(y) = \exp(y)$. The numerical determination of the integrating factor reduces, as above, to the method of computing the antiderivative from the integrand with subsequent calculation of the exponential function at the necessary points of the desired range. Multiplying Eq. (19) by $\mu(y)$, we proceed to the solution of the obtained exact differential equation

$$2xy^3 e^y\, dx + (3x^2y^2 + x^2y^3 + 1)e^y\, dy = 0.$$

To solve this equation, it is necessary to construct the function (potential) $F(x, y)$, such that

$$F_x(x, y) = 2xy^3 e^y, \tag{21}$$

$$F_y(x, y) = (3x^2y^2 + x^2y^3 + 1)e^y. \tag{22}$$

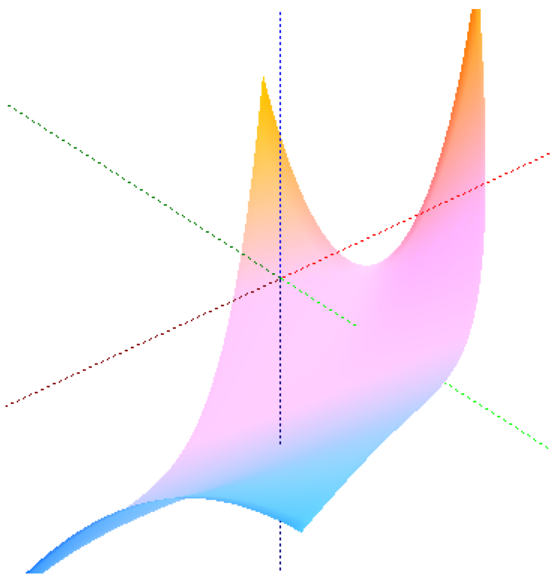
Substituting into formula (11) expressions (21), (22), and the expression of the mixed derivative

$$F_{xy}(x, y) = \left(6xy^2 + 2xy^3\right) e^y = 2xy^2(1 + y)e^y$$

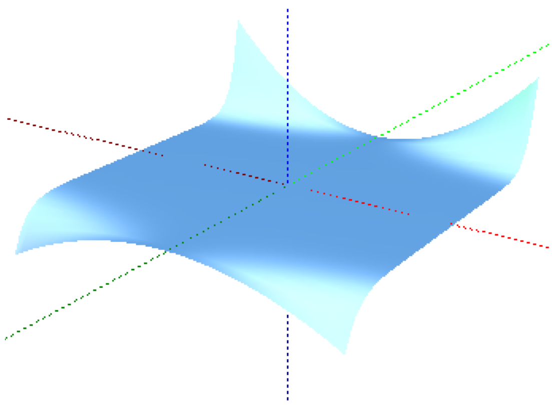
for each desired point $(x, y) \in R$ from the domain of solution existence under the given initial condition $y(0) = 0$, $x_0 = 0$, $y_0 = 0$, we construct the approximating surface of the potential (23) (Fig. 4).

The exact solution has the form

$$F(x, y) = \left(x^2y^3 + 1\right) e^y. \tag{23}$$



a



b

Fig. 4. The calculated potential surface an the error of its calculation: *a* — potential surface, the interval from −1.0 to +1.0 is shown; *b* — deviation from the exact surface $\left(F(x, y) - \left(x^2y^3 + 1\right) e^y\right) \cdot 10^{11}$ (color online)

Conclusion

The approximation of functions by the Chebyshev polynomials is optimal in L_∞ metric and near-optimal in L_2-norm metric [19]. Using the collocation method to calculate the coefficients of expansion in Chebyshev polynomials makes it possible to obtain a high approximation accuracy with a small number of terms in the series. The use of three-term relations provides high speed and accuracy of calculation of Chebyshev polynomials at arbitrary points of the solution definition interval.

Due to the discrete orthogonality of Chebyshev matrices, the Chebyshev collocation method on Gauss – Lobatto grids practically reduces the calculation of interpolation coefficients to multiplying the matrix by the vector of values of the interpolated function. The use of integration and differentiation matrices reduces nonlinear operations (integration and differentiation) to algebraic multiplication of sparse matrices by vectors. Numerical experiments demonstrate a decrease in computational costs by orders of magnitude compared to traditional methods for solving ODEs.

The method of integrating factors representing the solution as a product of functions is a variant of the method of separation of variables. This is a more interesting approach compared to presenting the solution as an expansion of the desired function $f(x)$ into a series and finding expansion coefficients by the method of least squares in one form or another (Bubnov – Petrov – Galerkin), since the product is a more complex construction, which offers more diverse possibilities when searching for the desired solution.

The approach to the numerical solution of ODEs based on integrating factors adds new possibilities compared to simple traditional Runge – Kutta methods. The speed and accuracy of the solution procedure sharply increases due to the use of a global approximation of the solution over the entire interval. The technique for solving first-order ODEs is simplified and generalized [13]. In many cases, the integrating factor method makes it possible to reduce ODEs with separable variables to exact ODEs and restore the desired potential with high accuracy.

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