Asymptotics of Solutions of Some Integral Equations Connected with Differential Systems with a Singularity

M. Yu. Ignatiev

Mikhail Yu. Ignatiev, https://orcid.org/0000-0002-4354-9197, Saratov State University, 83 Astrakhanskaya St., Saratov 410012, Russia, mikkieram@gmail.com, ignatievmu@info.sgu.ru

Our studies concern some aspects of scattering theory of the singular differential systems $y' - x^{-1}Ay - q(x)y = \rho By$, $x > 0$ with $n \times n$ matrices $A, B, q(x), x \in (0, \infty)$, where A, B are constant and ρ is a spectral parameter. We concentrate on investigation of certain Volterra integral equations with respect to tensor-valued functions. The solutions of these integral equations play a central role in construction of the so-called Weyl-type solutions for the original differential system. Actually, the integral equations provide a method for investigation of the analytical and asymptotical properties of the Weyl-type solutions while the classical methods fail because of the presence of the singularity. In the paper, we consider the important special case when q is smooth and $q(0) = 0$ and obtain the classical-type asymptotical expansions for the solutions of the considered integral equations as $\rho \to \infty$ with $o\left(\rho^{-1}\right)$ rate remainder estimate. The result allows one to obtain analogous asymptotics for the Weyl-type solutions that play in turn an important role in the inverse scattering theory.

Keywords: differential systems, singularity, integral equations, asymptotical expansions.

Received: 26.06.2019 / Accepted: 01.07.2019 / Published: 02.03.2020

DOI: https://doi.org/10.18500/1816-9791-2020-20-1-17-28

INTRODUCTION

Our studies concern some aspects of scattering theory of the differential systems

$$y' - x^{-1}Ay - q(x)y = \rho By, \quad x > 0 \tag{1}$$

with $n \times n$ matrices $A, B, q(x), x \in (0, \infty)$, where A, B are constant and ρ is a spectral parameter.

Differential equations with coefficients having non-integrable singularities at the end or inside the interval often appear in various areas of natural sciences and engineering. For $n = 2$, there exists an extensive literature devoted to different aspects of spectral theory of the radial Dirac operators, see, for instance [1–5].

Systems of the form (1) with $n > 2$ and arbitrary complex eigenvalues of the matrix B appear to be considerably more difficult for investigation even in the "regular" case $A = 0$ [6]. Some difficulties of principal matter also appear due to the presence of the singularity. Whereas the "regular" case $A = 0$ has been studied fairly completely to date [6–8], for the system (1) with $A \neq 0$ there are no similar general results.

The important role in scattering theory is played by a certain distinguished basis of generalized eigenfunctions for (1) (the so-called *Weyl-type solutions*, see, for instance [9]). In the presence of the singularity construction and investigation of this basis encounters some difficulties which do not appear in the "regular" case $A = 0$. In particular, one can not use the auxiliary Cauchy problems with the initial conditions at $x = 0$. The approach presented in [10] (see also [11] and references therein) for the scalar differential operators

$$\ell y = y^{(n)} + \sum_{j=0}^{n-2} \left(\frac{\nu_j}{x^{n-j}} + q_j(x) \right) y^{(j)} \tag{2}$$

is based on using some special solutions of the equation $\ell y = \lambda y$ that also satisfy certain Volterra integral equations. This approach assumes some additional decay condition for the coefficients $q_j(x)$ as $x \to 0$, moreover, the required decay rate depends on eigenvalues of the matrix A. In this paper, we do not impose any additional restrictions of such a type. Instead, we use a modification of the approach first presented in [12] for the higher-order differential operators with *regular* coefficients on the whole line and recently adapted for differential systems of the form (1) on the semi-axis in [9].

In brief outline the approach can be described as follows. We consider some auxiliary systems with respect to the functions with values in the exterior algebra $\wedge\mathbb{C}^n$. Our study of these auxiliary systems centers on two families of their solutions that also satisfy some asymptotical conditions as $x \to 0$ and $x \to \infty$ respectively, and can be constructed as solutions of certain *Volterra* integral equations. As in [12] we call these distinguished tensor solutions the *fundamental tensors*. The main difference from the above-mentioned method used in [10] is that we use the integral equations to construct the fundamental tensors rather than the solutions for the original system. Since each of the fundamental tensors has minimal growth (as $x \to 0$ or $x \to \infty$) among solutions of the same auxiliary system, this step does not require any decay of $q(x)$ as $x \to 0$.

Construction and properties of the fundamental tensors were considered in details in our paper [9] provided that $q(\cdot)$ is absolutely continuous and both q, q' are integrable on the semi-axis $(0, \infty)$. In this paper, we consider the important special case $q(0) = 0$ and obtain the classical-type asymptotical expansions for the fundamental tensors as $\rho \to \infty$ with $o\left(\rho^{-1}\right)$ rate remainder estimate.

1. ASSUMPTIONS AND NOTATIONS. FORMULATIONS OF THE RESULTS

We are to discuss first the unperturbed system:

$$y' - x^{-1}Ay = \rho By \tag{3}$$

and its particular case corresponding to the value $\rho = 1$ of the spectral parameter

$$y' - x^{-1}Ay = By \tag{4}$$

but to *complex* (in general) values of x.

Assumption 1. Matrix A is off-diagonal. The eigenvalues $\{\mu_j\}_{j=1}^n$ of the matrix A are distinct and such that $\mu_j - \mu_k \notin \mathbb{Z}$ for $j \neq k$, moreover, $\mathrm{Re}\mu_1 < \mathrm{Re}\mu_2 < \cdots < \mathrm{Re}\mu_n$, $\mathrm{Re}\mu_k \neq 0$, $k = \overline{1, n}$.

Assumption 2. $B = \mathrm{diag}\,(b_1, \ldots, b_n)$, the entries $b_1, \ldots, b_n$ are nonzero distinct points on the complex plane such that $\sum\limits_{j=1}^n b_j = 0$ and such that any 3 points are noncolinear.

Under Assumption 1 system (4) has the fundamental matrix $c(x) = (c_1(x), \ldots, c_n(x))$, where

$$c_k(x) = x^{\mu_k}\hat{c}_k(x),$$

$\det c(x) \equiv 1$ and all $\hat{c}_k(\cdot)$ are entire functions, $\hat{c}_k(0) = \mathfrak{h}_k$, $\mathfrak{h}_k$ is an eigenvector of the matrix A corresponding to the eigenvalue μ_k. We define $C_k(x, \rho) := c_k(\rho x)$, $x \in (0, \infty)$,

$\rho \in \mathbb{C}$. We note that the matrix $C(x,\rho)$ is a solution of the unperturbed system (3) (with respect to x for the given spectral parameter ρ).

Let Σ be the following union of lines through the origin in $\mathbb{C}$:

$$\Sigma = \bigcup_{(k,j):j\neq k} \{z : \mathrm{Re}(zb_j) = \mathrm{Re}(zb_k)\}.$$

By virtue of Assumption 2 for any $z \in \mathbb{C}\setminus\Sigma$ there exists the ordering $R_1,\dots,R_n$ of the numbers $b_1,\dots,b_n$ such that $\mathrm{Re}(R_1 z) < \mathrm{Re}(R_2 z) \dots < \mathrm{Re}(R_n z)$. Let $\mathscr{S}$ be a sector $\{z = r\exp(i\gamma), r \in (0,\infty), \gamma \in (\gamma_1,\gamma_2)\}$ lying in $\mathbb{C}\setminus\Sigma$. Then [13] the system (4) has the fundamental matrix $e(x) = (e_1(x),\dots,e_n(x))$ which is analytic in $\mathscr{S}$, continuous in $\overline{\mathscr{S}}\setminus\{0\}$ and admits the asymptotics:

$$e_k(x) = \mathrm{e}^{xR_k}(\mathfrak{f}_k + x^{-1}\eta_k(x)), \quad \eta_k(x) = O(1), \quad x\to\infty, \quad x \in \overline{\mathscr{S}},$$

where $(\mathfrak{f}_1,\dots,\mathfrak{f}_n) = \mathfrak{f}$ is a permutation matrix such that $(R_1,\dots,R_n) = (b_1,\dots,b_n)\mathfrak{f}$. We define $E(x,\rho) := e(\rho x)$.

Everywhere below we assume that the following additional condition is satisfied.

Condition 1. For all $k = \overline{2,n}$ the numbers

$$\Delta_k^0 := \det(e_1(x),\dots,e_{k-1}(x), c_k(x),\dots,c_n(x))$$

are not equal to 0.

Under Condition 1 the system (4) has the fundamental matrix $\psi^0(x) = (\psi_1^0(x),\dots,\psi_n^0(x))$ which is analytic in $\mathscr{S}$, continuous in $\overline{\mathscr{S}}\setminus\{0\}$ and admits the asymptotics:

$$\psi_k^0(xt) = \exp(xtR_k)(\mathfrak{f}_k + o(1)), \quad t\to\infty, \quad x\in\mathscr{S}, \quad \psi_k^0(x) = O(x^{\mu_k}), \quad x\to 0.$$

We define $\Psi^0(x,\rho) := \psi^0(\rho x)$. As above, we note that the matrices $E(x,\rho)$, $\Psi^0(x,\rho)$ solve (3).

In the sequel we use the following notations:

– $\{\mathfrak{e}_k\}_{k=1}^n$ is the standard basis in $\mathbb{C}^n$;

– $\mathscr{A}_m$ is the set of all ordered multi-indices $\alpha = (\alpha_1,\dots,\alpha_m)$, $\alpha_1 < \alpha_2 < \dots < \alpha_m$, $\alpha_j \in \{1,2,\dots,n\}$;

– for a sequence $\{u_j\}$ of vectors and a multi-index $\alpha = (\alpha_1,\dots,\alpha_m)$ we define $u_\alpha := u_{\alpha_1}\wedge\dots\wedge u_{\alpha_m}$;

– for a numerical sequence $\{a_j\}$ and a multi-index α we define $a_\alpha := \sum_{j\in\alpha} a_j$, $a^\alpha := \prod_{j\in\alpha} a_j$;

– for a multi-index α the symbol α' denotes the ordered multi-index that complements α to $(1,2,\dots,n)$;

– for $k = \overline{1,n}$ we denote

$$\overrightarrow{a}_k := \sum_{j=1}^k a_j, \quad \overleftarrow{a}_k := \sum_{j=k}^n a_j, \quad \overrightarrow{a}^k := \prod_{j=1}^k a_j, \quad \overleftarrow{a}^k := \prod_{j=k}^n a_j.$$

We note that Assumptions 1, 2 imply, in particular, $\sum_{k=1}^n \mu_k = \sum_{k=1}^n R_k = 0$ and therefore for any multi-index α one has $R_{\alpha'} = -R_\alpha$ and $\mu_{\alpha'} = -\mu_\alpha$;

– the symbol $V^{(m)}$, where V is $n \times n$ matrix, denotes the operator acting in $\wedge^m \mathbb{C}^n$ so that for any vectors $u_1, \dots, u_m$ the following identity holds:

$$V^{(m)}(u_1 \wedge u_2 \wedge \dots \wedge u_m) = \sum_{j=1}^{m} u_1 \wedge u_2 \wedge \dots \wedge u_{j-1} \wedge V u_j \wedge u_{j+1} \wedge \dots \wedge u_m;$$

– if $h \in \wedge^n \mathbb{C}^n$ then $|h|$ is a number such that $h = |h| \mathfrak{e}_1 \wedge \mathfrak{e}_2 \wedge \dots \wedge \mathfrak{e}_n$;

– for $h \in \wedge^m \mathbb{C}^n$ we set $\|h\| := \sum\limits_{\alpha \in \mathscr{A}_m} |h_\alpha|$, where $\{h_\alpha\}$ are the coefficients from the expansion $h = \sum\limits_{\alpha \in \mathscr{A}_m} h_\alpha \mathfrak{e}_\alpha$.

We use the same notation $L_p(a,b)$ for all the spaces of the form $L_p((a,b), \mathscr{E})$, where $\mathscr{E}$ is a finite-dimensional space. The notation $C[a,b]$ for the spaces of continuous functions will be used in a similar way.

Everywhere below the symbol $\mathscr{S}$ denotes some (arbitrary) open sector with the vertex at the origin lying in $\mathbb{C} \setminus \Sigma$.

For each fixed $\rho \in \overline{\mathscr{S}} \setminus \{0\} =: \mathscr{S}'$ we consider the following Volterra integral equations ($k = \overline{1,n}$):

$$Y(x) = T_k^0(x,\rho) + \int_0^x G_{n-k+1}(x,t,\rho) \left(q^{(n-k+1)}(t) Y(t) \right) dt, \tag{5}$$

$$Y(x) = F_k^0(x,\rho) - \int_x^\infty G_k(x,t,\rho) \left(q^{(k)}(t) Y(t) \right) dt, \tag{6}$$

where

$$T_k^0(x,\rho) := C_k(x,\rho) \wedge \dots \wedge C_n(x,\rho), \tag{7}$$

$$F_k^0(x,\rho) := E_1(x,\rho) \wedge \dots \wedge E_k(x,\rho) = \Psi_1^0(x,\rho) \wedge \dots \wedge \Psi_k^0(x,\rho) \tag{8}$$

and $G_m(x,t,\rho)$ is an operator acting in $\wedge^m \mathbb{C}^n$ as follows:

$$G_m(x,t,\rho) f = \sum_{\alpha \in \mathscr{A}_m} \sigma_\alpha \left| f \wedge C_{\alpha'}(t,\rho) \right| C_\alpha(x,\rho). \tag{9}$$

Here and below $\sigma_\alpha := |\mathfrak{h}_\alpha \wedge \mathfrak{h}_{\alpha'}|$.

For any $\rho \in \mathscr{S}'$ equations (5) and (6) were shown to have the unique solutions $T_k(x,\rho)$ and $F_k(x,\rho)$ respectively such that (see [9] for details):

$$\|T_k(x,\rho)\| \leqslant M \begin{cases} \left|(\rho x)^{\overleftarrow{\mu}_k}\right|, & |\rho x| \leqslant 1, \\ \left|\exp(\rho x \overleftarrow{R}_k)\right|, & |\rho x| > 1, \end{cases}$$

$$\|F_k(x,\rho)\| \leqslant M \begin{cases} \left|(\rho x)^{\overrightarrow{\mu}_k}\right|, & |\rho x| \leqslant 1, \\ \left|\exp(\rho x \overrightarrow{R}_k)\right|, & |\rho x| > 1. \end{cases}$$

We call the functions $F_k(x,\rho)$, $T_k(x,\rho)$ the *fundamental tensors*. Note that the fundamental tensors solve the auxiliary systems

$$Y' = Q^{(m)}(x,\rho) Y, \quad Q(x,\rho) := x^{-1} A + \rho B + q(x) \tag{10}$$

with $m = k$ and $m = n - k + 1$.

We note that the tensors $\{E_\alpha(x,\rho)\}_{\alpha\in\mathscr{A}_m}$ form the fundamental system of solutions for the system (10) in the "unperturbed" case. Therefore, the following representation holds:

$$T_k^0(x,\rho) = \sum_{\alpha\in\mathscr{A}_{n-k+1}} T_{k\alpha}^0 E_\alpha(x,\rho) \tag{11}$$

with x-independent coefficients $T_{k\alpha}^0$. Taking into account the special construction of the fundamental matrices $C(x,\rho)$, $E(x,\rho)$ one can conclude that the coefficients $T_{k\alpha}^0$ do not depend on ρ as well.

The $G_m(x,t,\rho)$ terms in equations (5), (6) are actually the Green operator functions for the nonhomogeneous systems:

$$Y' = Q^{(m)}(x,\rho)Y + f(x).$$

In order to construct them one can use variuos fundamental systems of solutions of the unperturbed system (3). In particular the following representations hold:

$$G_m(x,t,\rho)f = \sum_{\alpha\in\mathscr{A}_m} \chi_\alpha \left| f\wedge\Psi_{\alpha'}^0(t,\rho)\right| \Psi_\alpha^0(x,\rho) = \sum_{\alpha\in\mathscr{A}_m} \chi_\alpha \left| f\wedge E_{\alpha'}(t,\rho)\right| E_\alpha(x,\rho). \tag{12}$$

Here and below $\chi_\alpha := |\mathfrak{f}_\alpha \wedge \mathfrak{f}_{\alpha'}|$.

In the paper, we study the asymptotical behavior of the fundamental tensors for $\rho\to\infty$. In [9] the following expansions were obtained:

$$T_k(x,\rho) = T_k^0(x,\rho) + O\left(\rho^{-\varepsilon}\exp\left(\rho x \overleftarrow{R}_k\right)\right), \quad \varepsilon\in(0,1),$$
$$F_k(x,\rho) = F_k^0(x,\rho) + O\left(\rho^{-1}\exp\left(\rho x \overrightarrow{R}_k\right)\right)$$

for any fixed $x\in(0,\infty)$ and $\rho\to\infty$, $\rho\in\mathscr{S}'$. We show that under the additional condition $q(0)=0$ more detailed expansion can be obtained.

Let $W_0(\xi)$ be the function defined as follows:

$$W_0(\xi) = (1-|\xi|)\xi + |\xi|^2, \quad |\xi|\leqslant 1, \quad W_0(\xi) := \left(W_0\left(\xi^{-1}\right)\right)^{-1}, \quad |\xi|>1.$$

Notice that $W_0(\xi)$ is continuous in $\xi\in\mathbb{C}$, never vanishes for nonzero ξ and admits the estimate:

$$M_1|\xi| \leqslant |W_0(\xi)| \leqslant M_2|\xi|$$

for all $\xi\in\mathbb{C}$. Moreover, we have $W_0(\xi)=1$ if $|\xi|=1$ and the asymptotics $W_0(\xi)=\xi(1+o(1))$ hold as $\xi\to 0$ and $\xi\to\infty$.

We introduce the following weight functions:

$$W_k(\xi) := \begin{cases} W_0\left(\xi^{\mu_k}\right)\exp(R_k\xi), & |\xi|\leqslant 1, \\ \exp(R_k\xi), & |\xi|>1. \end{cases}$$

From the definition and the above-mentioned properties of $W_0(\cdot)$ it follows that the weight functions $W_k(\cdot), k=\overline{1,n}$ are all continuous in $\mathscr{S}'$, never vanish and admit the asymptotics $W_k(\xi)=\xi^{\mu_k}(1+o(1))$ as $\xi\to 0$. We define

$$\tilde{F}_k(x,\rho) := (\overrightarrow{W}^k(\rho x))^{-1}F_k(x,\rho), \quad \tilde{T}_k(x,\rho) := (\overleftarrow{W}^k(\rho x))^{-1}T_k(x,\rho).$$

Theorem 1. *Suppose that $q(\cdot)$ is an absolutely continuous off-diagonal matrix function such that $q(0) = 0$. Denote by $\hat{q}_o(\cdot)$ the off-diagonal matrix function such that $[B, \hat{q}_o(x)] = -q(x)$ for all $x > 0$ (here $[\cdot, \cdot]$ denotes the matrix commutator). Define the diagonal matrix $d(x) = \operatorname{diag}(d_1(x), \ldots, d_n(x))$, where*

$$d_k(x) := \int\limits_x^\infty t^{-1} \left([\hat{q}_o(t), A]\right)_{kk} \, dt$$

and set $\hat{q}(x) := \hat{q}_o(x) + d(x)$.

Suppose that all the functions $q_{ij}(\cdot), q'_{ij}(\cdot)$ and $\tilde{q}_{ij}(\cdot)$, where $\tilde{q}(x) := \hat{q}'(x) + x^{-1}[\hat{q}(x), A]$ are from $X_p := L_1(0, \infty) \cap L_p(0, \infty)$, $p > 2$.

Then for each fixed $x > 0$ and $\rho \to \infty$, $\rho \in \mathscr{S}'$ the following asymptotics hold:

$$\rho(\tilde{T}_k(x, \rho) - \tilde{T}_k^0(x, \rho)) = d_{0k} \tilde{T}_k^0(x, \rho) + \sum_{\alpha, \beta \in \mathscr{A}_{n-k+1}} T_{k\beta}^0 g_{k\alpha\beta}(x) \exp(\rho x(R_\beta - \overleftarrow{R}_k)) \mathfrak{f}_\alpha + o(1),$$

$$\rho\left(\tilde{F}_k(x, \rho) - \tilde{F}_k^0(x, \rho)\right) = \sum_{\alpha \in \mathscr{A}_k} f_{k\alpha}(x) \mathfrak{f}_\alpha + o(1).$$

Here

$$d_{0k} = -\sigma_{\alpha^*(k)} \left| \left(d^{(n-k+1)}(0) \mathfrak{h}_{\alpha^*(k)}\right) \wedge \mathfrak{h}_{(\alpha^*(k))'} \right|,$$

$\alpha^*(k) := (k, \ldots, n)$ and the coefficients in the representations are defined as follows:

$$f_{k\alpha}(x) = \chi_\alpha \left| \left(\hat{q}^{(k)}(x) \mathfrak{f}_{\alpha_*(k)}\right) \wedge \mathfrak{f}_{\alpha'} \right|$$

for $\alpha \neq \alpha_*(k) := (1, \ldots, k)$,

$$f_{k, \alpha_*(k)}(x) = -\sum_{\alpha \in \mathscr{A}_k} \int\limits_x^\infty \chi_{\alpha_*(k)} \left| \left(q^{(k)}(t) \mathfrak{f}_\alpha\right) \wedge \mathfrak{f}_{\alpha'_*(k)} \right| \chi_\alpha \left| (\hat{q}^{(k)}(t) \mathfrak{f}_{\alpha_*(k)}) \wedge \mathfrak{f}_{\alpha'} \right| dt;$$

$$g_{k\alpha\beta}(x) = \chi_\alpha \left| \left(\hat{q}^{(n-k+1)}(x) \mathfrak{f}_\beta\right) \wedge \mathfrak{f}_{\alpha'} \right|$$

for $\beta \neq \alpha$,

$$g_{k\beta\beta}(x) = \sum_{\alpha \in \mathscr{A}_{n-k+1}} \int\limits_0^x \chi_\beta \left| \left(q^{(n-k+1)}(t) \mathfrak{f}_\alpha\right) \wedge \mathfrak{f}_{\beta'} \right| \chi_\alpha \left| (\hat{q}^{(n-k+1)}(t) \mathfrak{f}_\beta) \wedge \mathfrak{f}_{\alpha'} \right| dt.$$

2. PROOF OF THEOREM 1

We consider in details the function $T_k(x, \rho)$, for the function $F_k(x, \rho)$ similar arguments are valid.

For the function $\hat{T}_k(x, \rho) := \tilde{T}_k(x, \rho) - \tilde{T}_k^0(x, \rho)$ we have the representation $\hat{T}_k(\cdot, \rho) = (Id - \mathscr{K}(\rho))^{-1} v_k(\cdot, \rho)$, where $\mathscr{K}(\rho)$ is an operator of the form:

$$\left(\mathscr{K}(\rho) f\right)(x) := \int\limits_0^x \mathscr{G}_{n-k+1}(x, t, \rho) \left(q^{(n-k+1)}(t) f(t)\right) dt$$

acting in $L_\infty(0, T)$, $T \in (0, \infty)$ is arbitrary. Here and below

$$\mathscr{G}_{n-k+1}(x, t, \rho) := \frac{\overleftarrow{W}^k(\rho t)}{\overleftarrow{W}^k(\rho x)} G_{n-k+1}(x, t, \rho),$$

$$v_k(x,\rho) = \int_0^x \mathscr{G}_{n-k+1}(x,t,\rho) \left(q^{(n-k+1)}(t) \tilde{T}_k^0(t,\rho) \right) dt.$$

Let us consider first the function $v_k(x,\rho)$. From the identity:

$$\rho(q^{(n-k+1)}(t)T_k^0(t,\rho)) \wedge E_{\alpha'}(t,\rho) =$$
$$= \frac{d}{dt}\left((\hat{q}^{(n-k+1)}(t)T_k^0(t,\rho)) \wedge E_{\alpha'}(t,\rho) \right) - (\tilde{q}^{(n-k+1)}(t)T_k^0(t,\rho)) \wedge E_{\alpha'}(t,\rho),$$

where $\alpha \in \mathscr{A}_{n-k+1}$ is arbitrary it follows the relation:

$$\rho \int_{x_0}^x G_{n-k+1}(x,t,\rho) \left(q^{(n-k+1)}(t) T_k^0(t,\rho) \right) dt =$$
$$= G_{n-k+1}(x,t,\rho) \left(\hat{q}^{(n-k+1)}(t) T_k^0(t,\rho) \right)\Big|_{t=x_0}^{t=x} - \int_{x_0}^x G_{n-k+1}(x,t,\rho) \left(\tilde{q}^{(n-k+1)}(t) T_k^0(t,\rho) \right) dt.$$

Passing to the limits as $x_0 \to 0$ and taking into account that $\hat{q}_o(0) = 0$ we arrive at the relation:

$$\rho v_k(x,\rho) = d_{0k}\tilde{T}_k^0(x,\rho) + \sum_{\alpha \in \mathscr{A}_{n-k+1}} \chi_\alpha \left| (\hat{q}^{(n-k+1)}(x)\tilde{T}_k^0(x,\rho)) \wedge E_{\alpha'}(x,\rho) \right| E_\alpha(x,\rho) -$$
$$\int_0^x \mathscr{G}_{n-k+1}(x,t,\rho) \left(\tilde{q}^{(n-k+1)}(t) \tilde{T}_k^0(t,\rho) \right) dt. \tag{13}$$

Since $\tilde{q}_{jj} = 0$, $j = \overline{1,n}$, from (13) and [14] we obtain (in particular) the estimate:

$$\|v_k(\cdot,\rho)\|_{BC[0,\infty)} = O(\rho^{-1}), \quad \rho \in \mathscr{S}'. \tag{14}$$

In what follows if $V = V(x,\rho)$ is some matrix function then $\tilde{V}$ denotes the matrix function $\tilde{V}(x,\rho) := V(x,\rho)(W(\rho x))^{-1}$, where $W = \operatorname{diag}(W_1, \dots W_n)$. Since $\tilde{\Psi}^0(x,\rho)$ is continuous and bounded in $[0,\infty) \times \overline{\mathscr{S}}$ we have:

$$\|\mathscr{G}_{n-k+1}(x,t,\rho)\| \leqslant M, \quad 0 < t \leqslant x < \infty, \quad \rho \in \mathscr{S}' \tag{15}$$

with some absolute constant M.

Using the boundedness of $\mathscr{G}_{n-k+1}(x,t,\rho)$ one can obtain the estimate (see also the proof of [9, Theorem 3.1]):

$$\|\mathscr{K}^r(\rho)\| \leqslant M_0 \frac{M_1^r}{r!} \left(\int_0^T \|q(t)\| dt \right)^r,$$

where the norm $\|\mathscr{K}^r(\rho)\|$ assumes the norm of the operator acting in $L_\infty(0,T)$ for arbitrary $T > 0$ and the constants M_0, M_1 do not depend on T. This yields the estimate $\|(Id - \mathscr{K}(\rho))^{-1}\| = O(1)$ uniformly in $\rho \in \mathscr{S}'$. Thus (with taking into account (14)), we obtain the auxiliary prior estimate for $\hat{T}_k$:

$$\|\hat{T}_k(\cdot,\rho)\|_{L_\infty(0,T)} = O(\rho^{-1}), \rho \in \mathscr{S}' \tag{16}$$

for any $T > 0$.

In order to make a more detailed study we represent the operator $\mathscr{K}(\rho)$ in the form $\mathscr{K}(\rho) = \mathscr{K}_0(\rho) + \mathscr{K}_1(\rho)$, where:

$$\mathscr{K}_0(\rho)f(x) :=$$
$$= \theta^+(|\rho x| - 1) \sum_{\alpha \in \mathscr{A}_{n-k+1}} \chi_\alpha \int\limits_{|\rho|^{-1}}^{x} \exp(\rho(x-t)(R_\alpha - \overleftarrow{R}_k)) \left| \left(q^{(n-k+1)}(t)f(t)\right) \wedge \mathfrak{f}_{\alpha'} \right| \mathfrak{f}_\alpha \, dt.$$

Here and below the symbols $\theta^{\pm}(\cdot)$ denote the Heaviside step functions:

$$\theta^+(\xi) = \begin{cases} 0, & \xi \leqslant 0, \\ 1, & \xi > 0, \end{cases} \quad \theta^-(\xi) = \begin{cases} 1, & \xi \leqslant 0 \\ 0, & \xi > 0 \end{cases} = 1 - \theta^+(\xi).$$

Lemma 1. *Under the conditions of Theorem 1 one has the estimate* $\|\mathscr{K}_1(\rho)\| = O\left(\rho^{-1}\right)$.

Proof. We split the operator as follows: $\mathscr{K}_1 = \mathscr{K}_0^{(1)} + \mathscr{K}_1^{(1)} + \mathscr{K}_2^{(1)}$, where:

$$(\mathscr{K}_0^{(1)} f)(x) = \theta^-(|\rho x| - 1) \int_0^x \mathscr{G}_{n-k+1}(x,t,\rho) \left(q^{(n-k+1)}(t)f(t)\right) \, dt,$$
$$(\mathscr{K}_1^{(1)} f)(x) = \theta^+(|\rho x| - 1) \int_0^{|\rho|^{-1}} \mathscr{G}_{n-k+1}(x,t,\rho) \left(q^{(n-k+1)}(t)f(t)\right) \, dt.$$

By virtue of (15) we have:

$$\|\mathscr{K}_1^{(1)} f\| \leqslant M\|f\| \cdot \int_0^{|\rho|^{-1}} \|q(t)\| \, dt \leqslant M|\rho|^{-1}\|f\| \cdot \|q(\cdot)\|_{L_\infty(0,T)}.$$

Proceeding in a similar way and taking into account that $(\mathscr{K}_0^{(1)} f)(x) \neq 0$ only if $|\rho x| \leqslant 1$ one can obtain the similar estimate for $\|\mathscr{K}_0^{(1)} f\|$.

Let us consider $\mathscr{K}_2^{(1)}$. Using the representation (9) for $G_{n-k+1}(x,t,\rho)$, the asymptotics

$$E_\alpha(x,\rho) = \exp(\rho x R_\alpha)(\mathfrak{f}_\alpha + O\left((\rho x)^{-1}\right)),$$

which is uniform in $|\rho x| \geqslant 1$ and taking into account that $\mathrm{Re}(\rho(x-t)(R_\alpha - \overleftarrow{R}_k)) \leqslant 0$ for any $0 \leqslant t \leqslant x$, $\rho \in \mathscr{S}'$, $\alpha \in \mathscr{A}_{n-k+1}$ we obtain the estimate:

$$\theta^+(|\rho x| - 1)\theta^+(|\rho t| - 1)\theta^+(x-t) \Big\| \mathscr{G}_{n-k+1}(x,t,\rho) \left(q^{(n-k+1)}(t)f(t)\right) -$$
$$\sum_{\alpha \in \mathscr{A}_{n-k+1}} \chi_\alpha \exp(\rho(x-t)(R_\alpha - \overleftarrow{R}_k)) \left| \left(q^{(n-k+1)}(t)f(t)\right) \wedge \mathfrak{f}_{\alpha'} \right| \mathfrak{f}_\alpha \Big\| \leqslant \frac{M}{|\rho t|}\|q(t)\|$$

with some absolute constant M. Since under the conditions of Theorem 1 $t^{-1}q(t) \in L_1(0,\infty)$ the estimate above yields

$$\|\mathscr{K}_2^{(1)} f\| \leqslant M|\rho|^{-1}\|f\| \cdot \int_0^\infty t^{-1}\|q(t)\| \, dt$$

and therefore $\|\mathscr{K}_2^{(1)}\| = O(\rho^{-1})$. □

Lemma 2. *Under the conditions of Theorem 1 one has the estimate* $\|\mathscr{K}_0^2(\rho)\| = O\left(\rho^{-1}\right)$.

Proof. We have:

$$(\mathscr{K}_0^2 f)(x) = \theta^+(|\rho x|-1) \sum_{\alpha\in\mathscr{A}_{n-k+1}} \int_{|\rho|^{-1}}^{x} \exp(\rho(x-t)(R_\alpha - \overleftarrow{R}_k))\chi_\alpha \times$$
$$\times \left|\left(q^{(n-k+1)}(t)(\mathscr{K}_0 f)(t)\right) \wedge \mathfrak{f}_{\alpha'}\right| \mathfrak{f}_\alpha \, dt,$$
$$\chi_\alpha \left|\left(q^{(n-k+1)}(t)(\mathscr{K}_0 f)(t)\right) \wedge \mathfrak{f}_{\alpha'}\right| = \theta^+(|\rho t|-1)\times$$
$$\times \sum_{\beta\in\mathscr{A}_{n-k+1}} \chi_\beta \int_{|\rho|^{-1}}^{t} \exp(\rho(t-\tau)(R_\beta - \overleftarrow{R}_k)) \left|\left(q^{(n-k+1)}(\tau)f(\tau)\right) \wedge \mathfrak{f}_{\beta'}\right| Q_{\alpha\beta}(t)\, d\tau,$$

where $Q_{\alpha\beta}(t) := \chi_\alpha \left|\left(q^{(n-k+1)}(t)\mathfrak{f}_\beta\right) \wedge \mathfrak{f}_{\alpha'}\right|$.

Thus, we can rewrite:

$$(\mathscr{K}_0^2 f)(x) = \theta^+(|\rho x|-1) \sum_{\alpha\beta\in\mathscr{A}_{n-k+1}} \int_{|\rho|^{-1}}^{x} \left|\left(q^{(n-k+1)}(\tau)f(\tau)\right) \wedge \mathfrak{f}_{\beta'}\right| H_{\alpha\beta}(x,\tau,\rho)\, d\tau,$$

where:

$$H_{\alpha\beta}(x,\tau,\rho) = \int_\tau^x Q_{\alpha\beta}(t) \exp(\rho(x-t)(R_\alpha - \overleftarrow{R}_k) + \rho(t-\tau)(R_\beta - \overleftarrow{R}_k))\mathfrak{f}_\alpha\, dt.$$

We notice again that $\operatorname{Re}(\rho(x-t)(R_\alpha - \overleftarrow{R}_k) + \rho(t-\tau)(R_\beta - \overleftarrow{R}_k)) \leqslant 0$ for any $0 \leqslant \tau \leqslant t \leqslant x$, $\rho \in \mathscr{S}'$, $\alpha,\beta \in \mathscr{A}_{n-k+1}$. Moreover, under the conditions of Theorem 1 $Q_{\alpha\beta}(\cdot)$ are absolutely continuous and $Q_{\alpha\beta}(t) \equiv 0$ if $\alpha = \beta$. This yields the estimate

$$\theta^+(|\rho\tau|-1)H_{\alpha\beta}(x,\tau,\rho) = O(\rho^{-1}),$$

which is uniform in $0 \leqslant \tau \leqslant x$, $\rho \in \mathscr{S}'$. The estimate implies the required assertion. □

Proof of Theorem 1. We have $\hat{T}_k(\cdot,\rho) = v_k(\cdot,\rho) + \mathscr{K}(\rho)v_k(\cdot,\rho) + \mathscr{K}^2(\rho)\hat{T}_k(\cdot,\rho)$.

We note that

$$(\mathscr{K}(\rho)\tilde{T}_k^0(\cdot,\rho))(x) = \int_0^x \mathscr{G}_{n-k+1}(x,t,\rho)\left(q^{(n-k+1)}(t)\tilde{T}_k^0(t,\rho)\right) = v_k(x,\rho) = O(\rho^{-1})$$

uniformly for $\rho \in \mathscr{S}'$, $x \in (0,T)$.

This, prior estimate (16), (14) and Lemmas 1, 2 yield:

$$\hat{T}_k(\cdot,\rho) = v_k(\cdot,\rho) + \mathscr{K}_0(\rho)\omega_k(\cdot,\rho) + O\left(\rho^{-2}\right), \tag{17}$$

where

$$\rho\omega_k(x,\rho) = \sum_{\alpha\in\mathscr{A}_{n-k+1}} \chi_\alpha \left|(\hat{q}^{(n-k+1)}(x)\tilde{T}_k^0(x,\rho)) \wedge E_{\alpha'}(x,\rho)\right| E_\alpha(x,\rho) -$$
$$- \int_0^x \mathscr{G}_{n-k+1}(x,t,\rho)\left(\tilde{q}^{(n-k+1)}(t)\tilde{T}_k^0(t,\rho)\right) dt. \tag{18}$$

and the $O(\cdot)$ term assumes an estimate in $L_\infty(0,T)$ norm.

From [14, Theorem 1] and (18) we have:

$$\rho\omega_k(x,\rho) = \sum_{\alpha\in\mathscr{A}_{n-k+1}} \chi_\alpha \left|(\hat{q}^{(n-k+1)}(x)\tilde{T}_k^0(x,\rho))\wedge E_{\alpha'}(x,\rho)\right| E_\alpha(x,\rho)+o(1),$$

that yields:

$$\theta^+(|\rho t|-1)\rho\omega_k(t,\rho) =$$
$$= \theta^+(|\rho t|-1)\sum_{\alpha,\beta\in\mathscr{A}_{n-k+1}} T^0_{k\beta}\exp(\rho t(R_\beta - \overleftarrow{R}_k))\hat{Q}_{\alpha\beta}(t)\mathfrak{f}_\alpha + \rho^{-1}\hat{\omega}_k(t,\rho)+o(1),$$

where $\hat{Q}_{\alpha\beta}(t) = \chi_\alpha\left|(\hat{q}^{(n-k+1)}(t)\mathfrak{f}_\beta)\wedge\mathfrak{f}_{\alpha'}\right|$, the $o(\cdot)$ term assumes an estimate in $L_\infty(0,T)$ norm and $t\hat{\omega}_k(t,\rho)$ is uniformly bounded in $\{|\rho t|\geqslant 1\}$.

Under the conditions of Theorem 1 we have $t^{-1}q(t)\in L_1(0,\infty)$. This yields $\mathscr{K}_0(\rho)\hat{\omega}_k(\cdot,\rho) = O(1)$ and thus from the representation above we obtain:

$$(\mathscr{K}_0(\rho)\omega_k(\cdot,\rho))(x) = \rho^{-1}\theta^+(|\rho x|-1)\sum_{\alpha,\beta,\gamma\in\mathscr{A}_{n-k+1}}\chi_\gamma T^0_{k\beta}\int\limits_{|\rho|^{-1}}^{x}\exp(\rho(x-t)(R_\gamma-\overleftarrow{R}_k)+$$
$$+\rho t(R_\beta-\overleftarrow{R}_k))\hat{Q}_{\alpha\beta}(t)\left|\left(q^{(n-k+1)}(t)\mathfrak{f}_\alpha\right)\wedge\mathfrak{f}_{\gamma'}\right|\mathfrak{f}_\gamma\,dt + o(\rho^{-1}) = \rho^{-1}\theta^+(|\rho x|-1)\times$$
$$\times\sum_{\beta,\gamma\in\mathscr{A}_{n-k+1}} T^0_{k\beta}\int\limits_{|\rho|^{-1}}^{x}\exp(\rho(x-t)(R_\gamma-\overleftarrow{R}_k)+\rho t(R_\beta-\overleftarrow{R}_k))\tilde{Q}_{\gamma\beta}(t)\mathfrak{f}_\gamma\,dt+o(\rho^{-1}),$$

where:

$$\tilde{Q}_{\gamma\beta}(t) := \sum_{\alpha\in\mathscr{A}_{n-k+1}} Q_{\gamma\alpha}(t)\hat{Q}_{\alpha\beta}(t),\quad Q_{\gamma\alpha}(t) = \chi_\gamma\left|\left(q^{(n-k+1)}(t)\mathfrak{f}_\alpha\right)\wedge\mathfrak{f}_{\gamma'}\right|$$

and the $o(\cdot)$ term assumes an estimate in $L_\infty(0,T)$. Under the conditions of Theorem 1 the functions $Q_{\alpha\beta}$ и $\hat{Q}_{\alpha\beta}$ (for any pair of multi-indices α,β) are absolutely continuous. Therefore, we have for $\gamma\neq\beta$:

$$\int\limits_{|\rho|^{-1}}^{x}\exp(\rho(x-t)(R_\gamma-\overleftarrow{R}_k)+\rho t(R_\beta-\overleftarrow{R}_k))\tilde{Q}_{\gamma\beta}(t)\,dt = O(\rho^{-1}),$$

that yields:

$$(\mathscr{K}_0(\rho)\omega_k(q,\cdot,\rho))(x) =$$
$$=\rho^{-1}\theta^+(|\rho x|-1)\sum_{\beta\in\mathscr{A}_{n-k+1}}T^0_{k\beta}\exp(\rho x(R_\beta-\overleftarrow{R}_k))\int\limits_{|\rho|^{-1}}^{x}\tilde{Q}_{\beta\beta}(t)\,dt\mathfrak{f}_\beta+o(\rho^{-1}).$$

Substituting the obtained asymptotics to the representation (17) we arrive at:

$$\hat{T}_k(x,\rho) = v_k(x,\rho)+\rho^{-1}\theta^+(|\rho x|-1)\times$$
$$\times\sum_{\beta\in\mathscr{A}_{n-k+1}}T^0_{k\beta}\exp(\rho x(R_\beta-\overleftarrow{R}_k))\int\limits_{|\rho|^{-1}}^{x}\tilde{Q}_{\beta\beta}(t)\,dt\mathfrak{f}_\beta+o(\rho^{-1}). \qquad (19)$$

Here, as above, the $o(\cdot)$ term assumes an estimate in $L_\infty(0,T)$ norm. But all the terms in (19) are actually continuous with respect to $x \in (|\rho^{-1}|, T)$. This means that the expansion can be considered in point-wise sense as $\rho \to \infty$ while $x > 0$ is arbitrary fixed.

Now we notice that

$$\int_{|\rho|^{-1}}^{x} \tilde{Q}_{\beta\beta}(t)\,dt \to \int_{0}^{x} \tilde{Q}_{\beta\beta}(t)\,dt = g_{k\beta\beta}(x)$$

as $\rho \to \infty$. Then we use the representation (13) for $v_k(x,\rho)$ and thus we obtain the required asymptotics. $\square$

Acknowledgements: This work was supported by the Russian Science Foundation (project No. 17-11-01193).

References

1. Brunnhuber R., Kostenko A., Teschl G. Singular Weyl – Titchmarsh – Kodaira theory for one-dimensional Dirac operators. *Monatshefte für Mathematik*, 2014, vol. 174, pp. 515–547. DOI: https://doi.org/10.1007/s00605-013-0563-5
2. Albeverio S., Hryniv R., Mykytyuk Ya. Reconstruction of radial Dirac operators. *J. Math. Phys.*, 2007, vol. 48, iss. 4, 043501, 14 p. DOI: https://doi.org/10.1063/1.2709847
3. Albeverio S., Hryniv R., Mykytyuk Ya. Reconstruction of radial Dirac and Schŕodinger operators from two spectra. *J. Math. Anal. Appl.*, 2008, vol. 339, iss. 1, pp. 45–57. DOI: https://doi.org/10.1016/j.jmaa.2007.06.034
4. Serier F. Inverse Problems Inverse spectral problem for singular Ablowitz–Kaup–Newell–Segur operators on [0, 1]. *Inverse Problems*, 2006, vol. 22, no. 4, pp. 1457–1484. DOI: https://doi.org/10.1088/0266-5611/22/4/018
5. Gorbunov O. B., Shieh C.-T., Yurko V. A. Dirac system with a singularity in an interior point. *Applicable Analysis*, 2016, vol. 95, iss. 11, pp. 2397–2414. DOI: https://doi.org/10.1080/00036811.2015.1091069
6. Beals R., Coifman R. R. Scattering and inverse scattering for first order systems. *Comm. Pure Appl. Math.*, 1984, vol. 37, iss. 1, pp. 39–90. DOI: https://doi.org/10.1002/cpa.3160370105
7. Zhou X. Direct and inverse scattering transforms with arbitrary spectral singularities. *Comm. Pure Appl. Math.* 1989, vol. 42, iss. 7, pp. 895–938. DOI: https://doi.org/10.1002/cpa.3160420702
8. Yurko V. A. Inverse spectral problems for differential systems on a finite interval. *Results Math.*, 2005, vol. 48, iss. 3–4, pp. 371–386. DOI: https://doi.org/10.1007/BF03323374
9. Ignatyev M. Spectral analysis for differential systems with a singularity. *Results Math.*, 2017, vol. 71, iss. 3–4, pp. 1531–1555. DOI: https://doi.org/10.1007/s00025-016-0605-0
10. Yurko V. A. On higher-order differential operators with a singular point. *Inverse Problems*, 1993, vol. 9, no. 4, pp. 495–502. DOI: https://doi.org/10.1088/0266-5611/9/4/004
11. Fedoseev A. E. Inverse problems for differential equations on the half-line having a singularity in an interior point. *Tamkang Journal of Mathematics*, 2011, vol. 42, no. 3, pp. 343–354. DOI: https://doi.org/10.5556/j.tkjm.42.2011.879
12. Beals R., Deift P., Tomei C. *Direct and inverse scattering on the line*. Providence, Rhod Island, American Mathematical Society, 1988. 209 p.
13. Sibuya Yu. Stokes phenomena. *Bull. Amer. Math. Soc.*, 1977, vol. 83, no. 5, pp. 1075–1077.
14. Ignatiev M. Integral transforms connected with differential systems with a singularity. *Tamkang Journal of Mathematics*, 2019, vol. 50, no. 3, pp. 253–268. DOI: https://doi.org/10.5556/j.tkjm.50.2019.3353

Cite this article as:
Ignatiev M. Yu. Asymptotics of Solutions of Some Integral Equations Connected with Differential Systems with a Singularity. *Izv. Saratov Univ. (N. S.), Ser. Math. Mech. Inform.*, 2020, vol. 20, iss. 1, pp. 17–28. DOI: https://doi.org/10.18500/1816-9791-2020-20-1-17-28

УДК 517.984

Асимптотики решений некоторых интегральных уравнений, связанных с дифференциальными системами с особенностью

М. Ю. Игнатьев

Игнатьев Михаил Юрьевич, кандидат физико-математических наук, доцент кафедры математической физики и вычислительной математики, Саратовский национальный исследовательский государственный университет имени Н. Г. Чернышевского, Россия, 410012, г. Саратов, ул. Астраханская, д. 83, mikkieram@gmail.com, ignatievmu@info.sgu.ru

В работе изучаются некоторые аспекты теории рассеяния для сингулярных систем дифференциальных уравнений $y' - x^{-1}Ay - q(x)y = \rho By,\ x > 0$ со спектральным параметром ρ, где $A, B, q(x), x \in (0,\infty)$ — $n \times n$ матрицы, причем матрицы A, B постоянны. Основным предметом исследования являются некоторые вольтерровские интегральные уравнения относительно тензорно-значных функций. Решения этих уравнений играют центральную роль в построении так называемых решений типа Вейля для исходной системы дифференциальных уравнений. Поскольку классические методы при наличии особенности оказываются неприменимыми, изучение рассматриваемых интегральных уравнений становится в этом случае ключевым этапом исследования аналитических и асимптотических свойств решений типа Вейля. В данной работе мы рассматриваем важный частный случай, когда матрица-функция $q(\cdot)$ является гладкой и $q(0) = 0$. В этом случае для решений рассматриваемых интегральных уравнений удается получить асимптотические разложения при $\rho \to \infty$ с оценкой остаточного члена $o\left(\rho^{-1}\right)$. Полученный результат позволяет получить асимптотики для решений типа Вейля, играющие, в свою очередь, важную роль при исследовании обратной задачи рассеяния.

Ключевые слова: дифференциальные системы, особенности, интегральные уравнения, асимптотические разложения.

Поступила в редакцию: 26.06.2019 / Принята: 01.07.2019 / Опубликована: 02.03.2020

Благодарности. Работа выполнена при поддержке Российского научного фонда (проект № 17-11-01193).

Образец для цитирования:
Ignatiev M. Yu. Asymptotics of Solutions of Some Integral Equations Connected with Differential Systems with a Singularity [Игнатьев М. Ю. Асимптотики решений некоторых интегральных уравнений, связанных с дифференциальными системами с особенностью] // Изв. Сарат. ун-та. Нов. сер. Сер. Математика. Механика. Информатика. 2020. Т. 20, вып. 1. С. 17–28. DOI: https://doi.org/10.18500/1816-9791-2020-20-1-17-28