



МАТЕМАТИКА

An Inverse Spectral Problem for Sturm – Liouville Operators with Singular Potentials on Graphs with a Cycle

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This paper is devoted to the solution of inverse spectral problems for Sturm–Liouville operators with singular potentials from class W_2^{-1} on graphs with a cycle. We consider the lengths of the edges of investigated graphs as commensurable quantities. For the spectral characteristics, we take the spectra of specific boundary value problems and special signs, how it is done in the case of classical Sturm–Liouville operators on graphs with a cycle. From the spectra, we recover the characteristic functions using Hadamard's theorem. Using characteristic functions and specific signs from the spectral characteristics, we construct Weyl functions (m-function) on the edges of the investigated graph. We show that the specification of Weyl functions uniquely determines the coefficients of differential equation on a graph and we obtain a constructive procedure for the solution of an inverse problem from the given spectral characteristics. In order to study this inverse problem, the ideas of spectral mappings method are applied. The obtained results are natural generalizations of the well-known results of on solving inverse problems for classical differential operators.

Keywords: Sturm – Liouville operator, singular potential, graph with a cycle.

Received: 26.02.2019 / Accepted: 05.05.2019 /

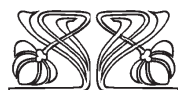
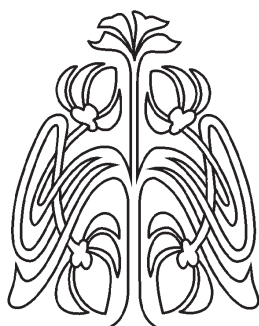
Published: 02.12.2019

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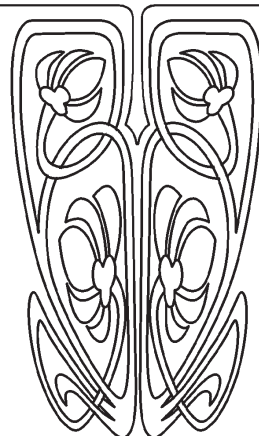
DOI: <https://doi.org/10.18500/1816-9791-2019-19-4-366-376>

INTRODUCTION

The paper concerns the theory of inverse spectral problems for differential operators on geometrical graphs. The inverse problem consists in recovering the potential from the given



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spectral characteristics. Differential operators on graphs are intensively studied by mathematicians in recent years and have applications in different branches of science and engineering. The inverse problem for the classical Sturm–Liouville operator on an *interval* has been studied comprehensively in the papers [1–4]. The case of inverse problem for Sturm–Liouville operators with potentials from class W_2^{-1} , which we call the *singular* potentials, on an *interval* was extensively studied in [5–7]. The inverse problems for a classical Sturm–Liouville operator on *graphs* was investigated in many papers [8–14]. The main result for such operators was obtained in [14], where the arbitrary graph has been considered. The case of inverse problem for Sturm–Liouville operators with singular potentials on *graphs* is more difficult for investigation, and nowadays there is only a number of papers in this area. The inverse problem on star-type graph with such type of potentials has been studied in [15]. Also, some specific types of graphs have been considered in papers [16, 17]. The inverse spectral problem for Sturm–Liouville operators with singular potentials on a graph with a cycle has not been studied yet because of the procedure of recovering characteristic functions from the spectra. In this paper we consider the solution of an inverse spectral problem for Sturm–Liouville differential operators with singular potentials on compact graphs with a cycle. As the spectral characteristics we consider the eigenvalues of specific boundary value problems and specific signs, as it is done in [9]. The lengths of the edges of a graph we consider as *commensurable* quantities. We provide a constructive procedure for the solution of inverse problem from given spectrums.

Let G be a graph with a set of vertices $\{v_j\}_{j=0}^p$ and a set of edges $\{e_j\}_{j=0}^p$, $e_j = [v_0, v_j]$, where edge e_0 generates a cycle. We suppose that the length of edge e_j is equal to $|e_j|$. We consider each edge e_j as a segment $[0, |e_j|]$ and parameterize it by the parameter $x \in [0, |e_j|]$. It is convenient for us to choose the orientation such that $x_j = |e_j|$ corresponds to the vertex v_0 . We consider lengths of edges e_j , $j = \overline{0, p}$, as commensurable quantities.

Function y on graph G is considered as $y = [y_j(x_j)]_{j=0}^p$, $x_j \in [0, |e_j|]$. Let $q = [q_j(x_j)]_{j=0}^p$ be a real-valued function on G such that $q_j \in W_2^{-1}[0, |e_j|]$, i.e. $q_j(x_j) = \sigma'_j(x_j)$, where the derivative is considered in the sense of distributions. We call function $\sigma = [\sigma_j(x_j)]_{j=0}^p$ the potential. The Sturm–Liouville differential operator on the edges e_j , $j = \overline{0, p}$ is defined by the following expression:

$$\ell_j y_j := -(y_j^{[1]})' - \sigma_j(x_j)y_j^{[1]} - \sigma_j^2(x_j)y_j,$$

where $y_j^{[1]} := y'_j - \sigma_j(x)y_j$ is a quasi-derivative, and

$$\text{dom}(\ell_j) = \{y_j \mid y_j \in W_2^1[0, |e_j|], y_j^{[1]} \in W_1^1[0, |e_j|], \ell_j y_j \in L_2[0, |e_j|]\}.$$

We consider the Sturm–Liouville equation on G :

$$(\ell_j y_j)(x_j) = \lambda y_j(x_j), \quad x_j \in (0, |e_j|), \quad y_j \in \text{dom}(\ell_j), \quad j = \overline{0, p}. \tag{1}$$

At the internal vertex v_0 we consider the following matching conditions:

$$y_0(0) = y_k(|e_k|), \quad k = \overline{0, p}, \quad \sum_{j=0}^p y_j^{[1]}(|e_j|) = y_0^{[1]}(0). \tag{2}$$

Let us consider the boundary value problem $L(G)$ for equation (1) with matching conditions (2) and boundary conditions

$$y_j^{[1]}(0) = 0, \quad j = \overline{1, p}. \tag{3}$$



We define the eigenvalues of $L(G)$ by Λ . We also consider the boundary value problem $L_k(G)$, $k = \overline{1, p}$, for equation (1) with matching conditions (2) and boundary conditions

$$y_j^{[1]}(0) = 0, \quad j = \overline{1, p} \setminus k, \quad y_k(0) = 0. \tag{4}$$

The symbol $\Delta(\lambda, L)$ denotes the *characteristic function* of some boundary value problem L . The eigenvalues of $L_k(G)$ we define by Λ_k . Let $\varphi_k = [\varphi_{kj}]_{j=1}^p$, $k = \overline{1, p}$, be the solutions of equation (1), satisfying (2) and boundary conditions:

$$\varphi_k^{[1]}(0) = \delta_{kj}, \tag{5}$$

where δ_{kj} is the Kronecker delta. Denote $M_k(\lambda) := \varphi_{kk}(0)$. The function $M_k(\lambda)$ is called the *Weyl function* for (1) with respect to the vertex v_k .

Graph G is partitioned into two parts by the vertex v_0 : $G = G_0 \cup T$, where G_0 is a star-type graph and T is a cycle, generated by edge e_0 . Let $Q(\lambda) := C_0(|e_0|, \lambda) - S_0^{[1]}(|e_0|, \lambda)$. Taking into account (2) и (3), one can show, that

$$\Delta(\lambda, L(T)) = C_0(|e_0|, \lambda) + S_0^{[1]}(|e_0|, \lambda) - 2. \tag{6}$$

Let $\ell(e_j)$, $j = \overline{0, p}$, be the boundary value problem for equation (1) on e_j with Dirichlet boundary condition in the end points of edge e_j , and let $\ell_j(e_j)$ be boundary value problem for equation (1) on e_j with boundary conditions $y_j^{[1]}(0) = 0$, $y_j(|e_j|) = 0$. Zeros of function $\Delta(\lambda, \ell(e_0))$ we denote as $\{z_n\}_{n \geq 1}$. Also we define $\omega_n := \text{sign} Q(z_n)$, $\Omega = \{\omega_n\}_{n \geq 1}$. The inverse problem is formulated as follows.

Inverse Problem 1. Given the Λ , Λ_k , $k = \overline{1, p}$, Ω , construct the potential q on G .

The paper is structured as follows. Section 1 contains some auxiliary propositions. Section 2 is devoted to the solution of so-called local inverse problems and a solution of the global inverse problem on a graph.

1. AUXILIARY PROPOSITIONS

Let $C_j(x_j, \lambda)$, $S_j(x_j, \lambda)$ be the solutions of equation (1) on edge e_j , $j = \overline{0, p}$ under initial conditions

$$C_j(0, \lambda) = S_j^{[1]}(0, \lambda) = 1, \quad C_j^{[1]}(0, \lambda) = S_j(0, \lambda) = 0. \tag{7}$$

As in the classical case [13] one can show that functions $M_j(\lambda)$, $j = \overline{1, p}$ are meromorphic in λ , namely:

$$M_j(\lambda) = -\frac{\Delta(\lambda, L_j(G))}{\Delta(\lambda, L(G))}, \quad M_0(\lambda) = -\frac{\Delta(\lambda, \ell_0(e_0))}{\Delta(\lambda, \ell(e_0))}. \tag{8}$$

Denote by B_r the class of Paley – Wiener functions of exponential type not greater than $r \in \mathbb{R}$, belonging to $L_2(\mathbb{R})$. It follows from [5–7] that

$$C_j(|e_j|, \lambda) = \cos \rho|e_j| + \zeta_{j,e}(\rho), \quad S_j(|e_j|, \lambda) = \frac{\sin \rho|e_j|}{\rho} + \frac{1}{\rho} \zeta_{j,o}(\rho), \tag{9}$$

where $\zeta_{j,e}(\rho) \in B_{|e_j|}$ are even functions and $\zeta_{j,o}(\rho) \in B_{|e_j|}$ are odd functions. Clearly,

$$\zeta_{j,o}(\rho) = \int_0^{|e_j|} K_o(t) \sin \rho t dt, \quad \zeta_{j,e}(\rho) = \int_0^{|e_j|} K_e(t) \cos \rho t dt, \quad K_o, K_e \in L_2(0, |e_j|).$$



Let $H_j, j = \overline{0, p}$, be the classes of functions, which are entire in ρ for all $x \in [0, |e_j|]$ and fixed potential σ_j , such that for $\eta_j(x, \rho, \sigma_j) \in H_j$ following conditions are valid:

- 1) $\eta_j(x, \rho, \sigma_j) = o(\exp(x|\operatorname{Im} \rho|))$ for $\rho \rightarrow \infty$ and any fixed $x \in [0, |e_j|]$ and $\sigma_j \in L_2[0, |e_j|]$;
- 2) $\eta_j(x, \cdot, \sigma_j) \in L_2(\gamma)$ for all $x \in [0, |e_j|]$, real τ and fixed $\sigma_j \in L_2[0, |e_j|]$, where

$$\gamma = \gamma(\tau) := (-\infty + i\tau, +\infty + i\tau);$$

- 3) $\eta_j(\cdot, \cdot, \sigma_j) \in L_2[0, |e_j|] \times \gamma$ and bounded uniformly on $[0, |e_j|] \times \gamma$ for any fixed real τ and $\sigma_j \in L_2[0, |e_j|]$;
- 4) $\eta_j(x, \rho, \sigma_j)$ depends continuously on the potential in the following sense: if $\sigma_{jn}(x) \rightarrow \sigma_j(x)$ in $L_2[0, |e_j|]$, then the corresponding $\eta_j(x, \rho, \sigma_{jn}) \in H_j$ converges to $\eta_j(x, \rho, \sigma_j) \in H_j$ uniformly on $[0, |e_j|] \times \gamma$ for all $\tau > \tau_0$ and

$$\max_{x \in [0, |e_j|]} \|\eta_j(x, \cdot, \sigma_{jn}) - \eta_j(x, \cdot, \sigma_j)\|_{L_2(\gamma)} \rightarrow 0.$$

Obviously, if $\eta_j(x, \rho, \sigma_j), \eta_j^*(x, \rho, \sigma_j) \in H_j$, then $\eta_j(x, \rho, \sigma_j) + \eta_j^*(x, \rho, \sigma_j) \in H_j$.

Define $A_\varepsilon(\tau_0) := \{\rho : \operatorname{Im} \rho \geq 0, \operatorname{dist}(\rho, K) > \varepsilon\}$, where $Z \subset \{\rho : 0 \leq \operatorname{Im} \rho \leq \tau_0\}$ is a countable set with a constrained number of points in $\operatorname{Re} \rho \in [t, t + 1], \operatorname{Im} \rho \in [0, \tau_0]$. Let K be the class of meromorphic functions, such that for $\kappa(\rho, \sigma) \in K$ following conditions are valid:

- 1) $\kappa(\rho, \sigma) = o(1)$ for $\rho \rightarrow \infty$ and fixed $\sigma \in L_2(G), \rho \in A_\varepsilon(\tau_0)$, where τ_0 depends on κ ;
- 2) $\kappa(\cdot, \sigma) \in L_2(\gamma)$ for all $\tau > \tau_0$ and fixed $\sigma \in L_2(G)$;
- 3) $\kappa(\rho, \sigma)$ depends continuously on σ , in the following sense: if $\sigma_{jn}(x) \rightarrow \sigma_j(x)$ in $L_2(G)$, then $\kappa(\rho, \sigma_n) \in K$ converges to $\kappa(\rho, \sigma) \in K$ uniformly on γ for all $\tau > \tau_0$ and

$$\lim_{n \rightarrow \infty} \|\kappa(\cdot, \sigma_n) - \kappa(\cdot, \sigma)\|_{L_2(\gamma)} \rightarrow 0.$$

Obviously, if $\kappa(\rho, \sigma), \kappa^*(\rho, \sigma) \in K$, then $\kappa(\rho, \sigma) + \kappa^*(\rho, \sigma) \in K$ and $\kappa(\rho, \sigma)\kappa^*(\rho, \sigma) \in K$. Define $[1] := 1 + \kappa(\rho), \kappa(\rho) \in K$. It follows from [5–7] that the following lemma holds.

Lemma 1. *Following representations are valid*

$$C_j(x, \lambda) = \cos \rho x + \eta_j(x, \rho, \sigma_j), \quad S_j(x, \lambda) = \frac{\sin \rho x}{\rho} + \frac{1}{\rho} \eta_j(x, \rho, \sigma_j). \quad (10)$$

Using (10), we obtain

$$\begin{aligned} C_j(|e_j|, \lambda) &= \cos \rho |e_j|[1], & S_j(|e_j|, \lambda) &= \frac{\sin \rho |e_j|}{\rho}[1], \\ C_j^{[1]}(|e_j|, \lambda) &= -\rho \sin \rho |e_j|[1], & S_j^{[1]}(|e_j|, \lambda) &= \cos \rho |e_j|[1]. \end{aligned} \quad (11)$$

We consider the solutions of equation (1):

$$\xi_j(x, \lambda) := C_j(x, \lambda) - i\rho S_j(x, \lambda), \quad E_j(x, \lambda) := C_j(x, \lambda) + i\rho S_j(x, \lambda), \quad j = \overline{0, p}.$$

Clearly, that $\langle \xi_j, E_j \rangle = 2i\rho$ and

$$\begin{aligned} \xi_j(|e_j|, \lambda) &= e^{-i\rho |e_j|}[1], & E_j(0, \lambda) &= e^{-i\rho |e_j|}[1], \\ \xi_j^{[1]}(|e_j|, \lambda) &= -i\rho e^{-i\rho |e_j|}[1], & E_j^{[1]}(0, \lambda) &= i\rho e^{-i\rho |e_j|}[1]. \end{aligned} \quad (12)$$



We consider the representation

$$\varphi_{jk}(x, \lambda) = A_{jk}(\lambda)\xi_j(x, \lambda) + B_{jk}(\lambda)E_j(x, \lambda), \quad j = \overline{0, p}. \tag{13}$$

Substituting (13) into (2) and (5), we obtain the system of linear equations with variables $A_{jk}(\lambda)$ and $B_{jk}(\lambda)$. The determinant of this system we define as $\Delta_E(\lambda, G)$.

Lemma 2. *Following representation is valid*

$$\begin{aligned} \Delta_E(\lambda, G) = & (-i\rho)^{p+1} \left[\sum_{j=0}^p \left(E_j^{[1]}(|e_j|, \lambda) + \xi_j^{[1]}(|e_j|, \lambda) \right) \prod_{i=0, i \neq j}^p \left(E_i(|e_i|, \lambda) + \xi_i(|e_i|, \lambda) \right) + \right. \\ & \left. + \left[i\rho \left(E_0(|e_0|, \lambda) - \xi_0(|e_0|, \lambda) \right) - 2i\rho \right] \prod_{i=1}^p \left(E_i(|e_i|, \lambda) + \xi_i(|e_i|, \lambda) \right) \right]. \end{aligned} \tag{14}$$

Proof. We define variable α in the internal vertex v_0 . Then $\Delta_E(\lambda, G)$ is the determinant of this system

$$\begin{aligned} -i\rho A_{jk} + i\rho B_{jk} &= \delta_{jk}, \quad j = \overline{1, p}, \\ \xi_j(|e_j|, \lambda) A_{jk} + E_j(|e_j|, \lambda) B_{jk} &= \alpha, \quad j = \overline{0, p}, \\ \sum_{j=0}^p (\xi_j^{[1]}(|e_j|, \lambda) A_{jk} + E_j^{[1]}(|e_j|, \lambda) B_{jk}) + i\rho A_{0k} - i\rho B_{0k} &= 0. \end{aligned}$$

Using Laplace expansion for rows, which defines the matching condition for edge e_0 , we obtain

$$\begin{aligned} \Delta_E(\lambda, G) = & (E_0(|e_0|, G) + \xi_0(|e_0|, G))\Delta_E(\lambda, G_0) + \\ & + (-i\rho)^{p+1} \prod_{j=1}^p \left(E_j(|e_j|, \lambda) + \xi_j(|e_j|, \lambda) \right) \left[\left(E_0^{[1]}(|e_0|, \lambda) + \xi_0^{[1]}(|e_0|, \lambda) \right) + \right. \\ & \left. - \left(\xi_0(|e_0|, \lambda) E_0^{[1]}(|e_0|, \lambda) - \xi_0^{[1]}(|e_0|, \lambda) E_0(|e_0|, \lambda) + i\rho \xi_0(|e_0|, \lambda) - i\rho E_0(|e_0|, \lambda) \right) \right], \end{aligned} \tag{15}$$

where G_0 is a star-type graph. Let us define the determinant of the system of linear equations by $d_m(\lambda)$

$$\left(E_j(|e_j|, \lambda) + \xi_j(|e_j|, \lambda) \right) a_j = \alpha, \quad j = \overline{m, p}, \quad \sum_{j=m}^p \left(E_j^{[1]}(|e_j|, \lambda) + \xi_j^{[1]}(|e_j|, \lambda) \right) a_j = 0$$

with variables $a_j, j = \overline{1, p}, \alpha$. We add each j -th column to $(j + p)$ -th column, $j = \overline{1, p}$, and get $\Delta_E(\lambda, G_0) = d_1(\lambda)$. Using Laplace expansion for $d_m(\lambda), m < p$, for the first row, we obtain

$$\begin{aligned} d_m(\lambda) = & \left(E_m(|e_m|, \lambda) + \xi_m(|e_m|, \lambda) \right) d_{m+1}(\lambda) + \\ & + \left(E_m^{[1]}(|e_m|, \lambda) + \xi_m^{[1]}(|e_m|, \lambda) \right) \prod_{i=m+1}^p \left(E_i(|e_i|, \lambda) + \xi_i(|e_i|, \lambda) \right). \end{aligned}$$



It is obvious, that $d_p(\lambda) = \left(E_m^{[1]}(|e_m|, \lambda) + \xi_m^{[1]}(|e_m|, \lambda) \right)$. Using mathematical induction, one can show, that for $1 \leq m \leq p$ following representations are valid:

$$\Delta_E(\lambda, G_0) = (i\rho)^{p+1} \sum_{j=1}^p \left(E_j^{[1]}(|e_j|, \lambda) + \xi_j^{[1]}(|e_j|, \lambda) \right) \prod_{i=1, i \neq j}^p \left(E_i(|e_i|, \lambda) + \xi_i(|e_i|, \lambda) \right).$$

Substitute this representation into (15), we obtain (14). □

Taking into account (12) and properties of the function $\kappa(\rho) \in K$, one can show

Corollary 1. Define $\theta := \left\{ \sum_{j=0}^p k_j |e_j|, k_j \in \{0, 1\} \right\}$, a $|G| := \sum_{j=0}^p |e_j|$. Then following representation is valid:

$$\Delta_E(\lambda, G) = (-i\rho)^{p+1} \sum_{l \in \theta} A_l(G) e^{-i\rho l} [1], \quad A_{|G|} \neq 0, \tag{16}$$

From [18] one can obtain the following lemma.

Lemma 3. For sufficiently large $|\rho|$, such that $\rho \in A(\tau_0)$, τ_0 is fixed, following estimate is valid

$$C_1 |\rho|^{p+1} e^{|\rho| \operatorname{Im} \rho} < |\Delta_E(\lambda, G)| < C_2 |\rho|^{p+1} e^{|\rho| \operatorname{Im} \rho}. \tag{17}$$

The following lemma describes the asymptotic representations of the solutions φ_k .

Lemma 4. For fixed $x \in [0, |e_j|]$ and for $\rho \in A(\tau_0)$, where τ_0 is fixed and $\rho \rightarrow \infty$, following estimates are valid:

$$\varphi_{jk}(x, \lambda) = O\left(\frac{1}{\rho} e^{-x \operatorname{Im} \rho}\right), \quad \varphi_{jk}^{[1]}(x, \lambda) = O\left(e^{-x \operatorname{Im} \rho}\right), \quad \hat{\varphi}_{jk}(x, \lambda) = \frac{1}{\rho} e^{i\rho x} \hat{\kappa}(\rho). \tag{18}$$

Proof. Using (13) by Cramer’s rule, we obtain

$$A_{jk}(\lambda) = \frac{d_{jk}(\lambda)}{\Delta_E(\lambda, G)}, \quad B_{jk}(\lambda) = \frac{d^{jk}(\lambda)}{\Delta_E(\lambda, G)}, \tag{19}$$

where $d_{jk}(\lambda)$ and $d^{jk}(\lambda)$ are determinants of the matrices, formed by replacing the corresponding column by the column of free terms.

Analogous to proof of Lemma 2, we obtain

$$\begin{aligned} d_{jk}(\lambda) &= -(i\rho)^{p+1} \sum_{l \in \Theta_j} B_l(G) e^{-i\rho l} [1], \quad B_{|G|-2|e_k|} \neq 0, \\ d^{jk}(\lambda) &= -(i\rho)^{p+1} \sum_{l \in \Theta} C_l(G) e^{-i\rho l} [1], \quad C_{|G|} \neq 0, \end{aligned} \tag{20}$$

$$\Theta := \left\{ \sum_{j=1}^p \theta_j |e_j| \mid \theta_j \in \{0, 1\}, j = \overline{1, p} \right\},$$

$$\Theta_k := \left\{ \sum_{j=1}^p \theta_j |e_j| \mid \theta_j \in \{0, 1\}, j = \overline{1, p} \setminus k, \theta_k = -1 \right\}.$$



Thus, we obtain

$$\begin{aligned} C_1|\rho|^p e^{(|G|-2|e_j|)\operatorname{Im}\rho} < |d_{jk}(\lambda)| < C_2|\rho|^p e^{(|G|-2|e_j|)\operatorname{Im}\rho}, \\ C_1|\rho|^p e^{|G|\operatorname{Im}\rho} < |d^{jk}(\lambda)| < C_1|\rho|^p e^{|G|\operatorname{Im}\rho}. \end{aligned} \tag{21}$$

Using (16) and (20), we obtain

$$\frac{d_{jk}(\lambda)}{\Delta_E(\lambda, G)} = -\frac{\sum_{l \in \Theta_j} B_l(G)e^{-i\rho l}}{\sum_{l \in \Theta} A_l(G)e^{-i\rho l}} [1], \quad \frac{d^{jk}(\lambda)}{\Delta_E(\lambda, G)} = -\frac{\sum_{l \in \Theta} C_l(G)e^{-i\rho l}}{\sum_{l \in \Theta} A_l(G)e^{-i\rho l}} [1] \tag{22}$$

for $\rho \in A(\tau_0)$.

Consequently, we get

$$\frac{d_{jk}(\lambda)}{\Delta_E(\lambda, G)} = A_{jk}^0 [1], \quad \frac{d^{jk}(\lambda)}{\Delta_E(\lambda, G)} = B_{jk}^0 [1], \tag{23}$$

where A_{jk}^0 and B_{jk}^0 are coefficients of an expansion in a fundamental system of solutions $\{\xi_j(x, \lambda), E_j(x, \lambda)\}$ in case $\sigma = 0$. Substitute (17) and (21) into (22), for sufficiently large $\rho \in A_\varepsilon$ we obtain following estimates:

$$|A_{jk}^0| \leq \frac{C}{\rho} e^{-2|e_j|\operatorname{Im}\rho}, \quad |B_{jk}^0| \leq \frac{C}{\rho}.$$

Thus, using (23), we obtain analogously, that for $\rho \rightarrow \infty$ and $\rho \in A(\tau_0)$ following estimates are valid:

$$A_{jk} = \frac{1}{\rho} O(e^{-2|e_j|\operatorname{Im}\rho}), \quad B_{jk} = O\left(\frac{1}{\rho}\right). \tag{24}$$

Substitute this estimates into (13), we obtain (18). □

Analogous to Lemma 2 and Corollary 1, one can prove

Lemma 5. *Following formulas are valid*

$$\begin{aligned} \Delta(\lambda, L(G)) &= \Delta(\lambda, \ell(e_0))\Delta(\lambda, L(Q)) + (\Delta(\lambda, L(T)) + 1) \prod_{k=1}^p \Delta(\lambda, \ell_k(e_k)), \\ \Delta_j(\lambda, L(G)) &= \Delta(\lambda, \ell(e_0))\Delta_j(\lambda, L(Q)) + \\ &+ (\Delta(\lambda, L(T)) + 1)\Delta(\lambda, \ell(e_j)) \prod_{k=\overline{1, p} \setminus \{j\}}^p \Delta(\lambda, \ell_k(e_k)), \end{aligned} \tag{25}$$

where $\Delta(\lambda, L(Q)) = \sum_{j=1}^p C_j^{[1]}(|e_j|, \lambda) \prod_{i=1, i \neq j}^p C_i(|e_i|, \lambda)$.

Corollary 2. *Following representation is valid*

$$\Delta(\lambda, L(G)) = (-1)^p \sum_{l \in \Theta} D_l(G)e^{-i\rho l} [1], \quad D_{|G|} \neq 0. \tag{26}$$



The eigenvalues Λ can be numbered as $\{\lambda_{nk}\}_{k=0, \mu_0-1, n \in \mathbb{N}} \cup \{\lambda_{nk}\}_{k=\overline{\mu_0, m}, n \in \mathbb{Z}}$, where $m \in \mathbb{N}$, μ_0 is a multiplicity of a zero eigenvalue of the boundary value problem $L(G)$ with zero potential. Analogously, the eigenvalues Λ_k can be numbered as $\{\lambda_{nkj}\}_{j=0, \mu_0^k-1, n \in \mathbb{N}} \cup \{\lambda_{nkj}\}_{j=\overline{\mu_0^k, m_k}, n \in \mathbb{Z}}$, where $m_k \in \mathbb{N}$, μ_0^k is a multiplicity of zero eigenvalue of the boundary value problem $L_k(G)$ with zero potential. From [19] it follows, that characteristic functions $\Delta(\lambda, L(G))$ and $\Delta(\lambda, L_k(G))$, $k = \overline{1, p}$, can be constructed from spectra by Hadamard’s theorem:

Theorem 1. *The specification of spectrums Λ and Λ_j uniquely determines the characteristic functions respectively by the formula*

$$\begin{aligned} \Delta(\lambda, L(G)) &= (-1)^{\mu_0} \frac{\partial^{\mu_0}}{\partial \lambda^{\mu_0}} \Delta_0(\lambda, L(G)) \Big|_{\lambda=0} \prod_{n=0}^{\infty} \prod_{k=0}^{\mu_0-1} \frac{\lambda_{nk} - \lambda}{\lambda_{nk}^{01}} \prod_{n=-\infty}^{\infty} \prod_{k=\mu_0}^m \frac{\lambda_{nk} - \lambda}{\lambda_{nk}^{01}}, \\ \Delta(\lambda, L_j(G)) &= (-1)^{\mu_0^j} \frac{\partial^{\mu_0^j}}{\partial \lambda^{\mu_0^j}} \Delta_0(\lambda, L_j(G)) \Big|_{\lambda=0} \prod_{n=0}^{\infty} \prod_{k=0}^{\mu_0^j-1} \frac{\lambda_{jnk} - \lambda}{\lambda_{jnk}^{01}} \prod_{n=-\infty}^{\infty} \prod_{k=\mu_0^j}^m \frac{\lambda_{jnk} - \lambda}{\lambda_{jnk}^{01}}. \end{aligned} \tag{27}$$

2. SOLUTION OF THE INVERSE PROBLEM

Fix e_k , $k \in \overline{1, p}$ and consider the following auxiliary inverse problem.

Local inverse problems $IP(k, G)$: Given $M_k(\lambda)$, construct $\sigma_k(x)$, $x \in [0, |e_k|]$.

Everywhere below if a symbol α denotes an object, related to σ , then $\tilde{\alpha}$ will denote the analogous object, related to $\tilde{\sigma}$ and $\hat{\alpha} = \alpha - \tilde{\alpha}$. Using the properties of functions from class K , analogous to [15] one can prove the following theorem:

Theorem 2. *If $M_k(\lambda) \equiv \tilde{M}_k(\lambda)$, then $\sigma_k(x) \equiv \tilde{\sigma}_k(x)$ almost everywhere on $[0, |e_k|]$.*

In ρ -plane consider the contour $\gamma = \gamma(\tau) := (-\infty + i\tau, +\infty + i\tau)$, where $\tau > 0$ is such that

$$\inf\{\Lambda_k \cup \tilde{\Lambda}_k\} > -\tau^2.$$

Let Γ be the contour in λ -plane which is an image of γ under the mapping $\lambda = \rho^2$. Denote by D^+ the image of the half-plane $\{\text{Im } \rho > \tau\}$ and $D^- := C \setminus D^+$. Let $C_N := \{|\lambda| = (N+1/4)^2\}$ and $C_N^- := C_N \cap D^-$ be the contours with clockwise orientation. Denote $\Gamma_N = \Gamma \cap \text{int} C_N$, $\Gamma_N^- = \Gamma_N \cup C_N^-$. Denote $\theta^2 = \mu$. Define the functions

$$\begin{aligned} D_k(x, \lambda, \mu) &:= \frac{\langle C_k(x, \lambda), C_k(x, \mu) \rangle}{\lambda - \mu} = \int_0^x C_k(t, \lambda) C_k(t, \mu) dt, \\ \tilde{D}_k(x, \lambda, \mu) &:= \frac{\langle \tilde{C}_k(x, \lambda), \tilde{C}_k(x, \mu) \rangle}{\lambda - \mu} = \int_0^x \tilde{C}_k(t, \lambda) \tilde{C}_k(t, \mu) dt, \end{aligned}$$

$$r_k(x, \rho, \theta) := D_k(x, \lambda, \mu) \theta \hat{M}_k(\mu), \quad \tilde{r}_k(x, \rho, \theta) := \tilde{D}_k(x, \lambda, \mu) \theta \hat{M}_k(\mu).$$

Everywhere below we chose contour $\gamma(\tau)$ such that $\theta \hat{M}_e(\mu) \in L_2(\gamma)$. Analogous to [15], one can obtain the main equation

$$\Psi_k(x) = \tilde{H}_k(x) \Psi_k(x) + \tilde{F}_k(x), \tag{28}$$



where $\Psi_k(x, \rho) := C_k(x, \lambda) - \tilde{C}_k(x, \lambda)$,

$$\tilde{F}_k(x) := \frac{1}{2\pi i} \lim_{N \rightarrow \infty} \int_{\Gamma_N} \tilde{D}_k(x, \lambda, \mu) \hat{M}_k(\mu) \tilde{C}_k(x, \mu) d\mu, \tag{29}$$

and for all fixed $x \in [0, |e_k|]$

$$\tilde{H}_k(x)f(\rho) := \frac{1}{\pi i} \int_{\gamma} \tilde{r}_k(x, \rho, \theta) f(\theta) d\theta, \quad H_k(x)f(\rho) := \frac{1}{\pi i} \int_{\gamma} r_k(x, \rho, \theta) f(\theta) d\theta,$$

and $\tilde{H}_k(x)$ is a Hilbert – Schmidt operator in $L_2(\gamma)$. Also from [15] we obtain the validity of the following theorems:

Theorem 3. For each fixed $x \in [0, |e_k|]$ equation (28) is uniquely solvable in $L_2(\gamma)$.

Using the solution $\Psi_k(x, \rho)$ of the main equation (28), one can calculate the function $C_k(x, \lambda)$ and then construct $\sigma_k(x)$ according to the next theorem.

Theorem 4. The solution $\sigma_k(x)$ of the Problem $IP(k)$ can be found by the formula

$$\sigma_k(x) = -\frac{1}{\pi i} \int_{\Gamma} \tilde{C}_k(x, \mu) \hat{C}_k(x, \mu) \hat{M}_k(\mu) d\mu + \frac{1}{\pi i} \lim_{N \rightarrow \infty} \int_{\gamma_N} \rho \cos 2\rho x \hat{M}_k(\rho^2) d\rho, \tag{30}$$

where $\gamma_N = \gamma \cap \{\rho : |\rho|^2 = (N + 1/4)^2\}$.

Thus, the solution of the local inverse problem $IP(k)$ can be constructed by the following algorithm.

Algorithm 1. Given $M_k(\lambda)$

1. Take $\tilde{\sigma} = 0$ and calculate $\tilde{C}_k(x, \lambda)$, $\tilde{M}_k(\lambda)$, $\tilde{D}_k(x, \lambda, \mu)$ and $\tilde{r}_k(x, \rho, \theta)$.
2. Construct $\tilde{F}_k(x, \rho)$ by (29).
3. Find $\Psi_k(x, \rho)$ by solving the main equation (28) for each $x \in [0, |e_k|]$.
4. Construct $\sigma_k(x)$ using (30), where $\hat{C}_k(x, \lambda) = \Psi_k(x, \rho)$.

Using $\Delta(\lambda, \ell_0(e_0)) = -C_0(|e_0|, \lambda)$, $\Delta(\lambda, \ell(e_0)) = S_0(|e_0|, \lambda)$ and Lemma 1, analogous to [16], we obtain

$$M_0(\lambda) = \sum_{n=0}^{\infty} \frac{M_n}{\lambda - z_n}, \quad M_n = -\frac{\Delta(z_n, \ell_0(e_0))}{\dot{\Delta}(z_n, \ell(e_0))}, \tag{31}$$

where $\dot{\Delta}(\lambda, \ell(e_0)) = \frac{d}{d\lambda} \Delta(\lambda, \ell(e_0))$ and M_n is Weyl sequence. The solution of the inverse problem 1 can be constructed by the following algorithm.

Algorithm 2. Given $\{\lambda_n\}_{n \geq 0}$, $\{\lambda_{nk}\}_{n \geq 0}$, $k = \overline{1, p}$.

1. Construct $\Delta(\lambda, G)$, $\Delta_k(\lambda, G)$, $k = \overline{1, p}$ by (27). Find $M_k(\lambda)$ using (8). For edges e_j , $j = \overline{1, p}$, we find σ_j by solving local inverse problems by Algorithm 1.
2. Calculate $\Delta(\lambda, \ell(e_0))$ and $\Delta(\lambda, L(T))$ via (25).
3. Find zeros $\{z_n\}_{n \geq 1}$ of the function $\Delta(\lambda, \ell(e_0))$.
4. Define $D(\lambda) = \Delta(\lambda, L(T)) + 2$. Calculate $Q(z_n) = \omega_n \sqrt{D^2(z_n) - 4}$.
5. Calculate $\Delta(z_n, \ell_0(e_0)) = \frac{1}{2}(D(z_n) + Q(z_n))$.
6. Find M_n by (31).
7. Calculate $M_0(\lambda)$ via (31) and find σ_0 by solving local inverse problem on edge e_0 by Algorithm 1.

Acknowledgements: This work was supported in part by the Russian Ministry of Education and Science (project No. 1.1660.2017/4.6) and by the Russian Foundation for Basic Research (project No. 19-01-00102).



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Cite this article as:

Vasilev S. V. An Inverse Spectral Problem for Sturm–Liouville Operators with Singular Potentials on Graphs with a Cycle. *Izv. Saratov Univ. (N. S.), Ser. Math. Mech. Inform.*, 2019, vol. 19, iss. 4, pp. 366–376. DOI: <https://doi.org/10.18500/1816-9791-2019-19-4-366-376>

УДК 517.984

Обратная задача для операторов Штурма–Лиувилля с сингулярными потенциалами на графах с циклами

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В данной статье исследуются обратные спектральные задачи для дифференциальных операторов Штурма–Лиувилля с сингулярными потенциалами из класса W_2^{-1} на графе с циклом. Длины рёбер рассматриваемого графа мы будем считать соизмеримыми величинами. В качестве спектральных характеристик мы рассмотрим спектры некоторых краевых задач, а также специальные знаки, аналогично тому, как это сделано в случае классических операторов Штурма–Лиувилля, заданных на графе с циклом. Используя теорему Адамара, мы восстановим характеристические функции по заданным спектрам краевых задач. Применяя восстановленные характеристические функции, мы построим функции Вейля (так называемые m -функции) на рёбрах рассматриваемого графа. Мы покажем, что задание функций Вейля однозначно определяет коэффициенты дифференциального уравнения на исследуемом графе. Также мы получим конструктивную процедуру решения обратной задачи по заданным спектральным характеристикам. Для решения поставленной задачи в работе используются идеи метода спектральных отображений, применённого для решения обратной задачи для классических операторов Штурма–Лиувилля. Полученный результат является обобщением хорошо известных результатов для обратных задач для классических дифференциальных операторов.

Ключевые слова: оператор Штурма–Лиувилля, сингулярный потенциал, граф с циклом.

Поступила в редакцию: 26.02.2019 / Принята: 05.05.2019 / Опубликовано: 02.12.2019

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Благодарности. Работа выполнена при финансовой поддержке Минобрнауки РФ (проект № 1.1660.2017/4.6) и РФФИ (проект № 19-01-00102).

Образец для цитирования:

Vasilev S. V. An Inverse Spectral Problem for Sturm–Liouville Operators with Singular Potentials on Graphs with a Cycle [*Васильев С. В.* Обратная задача для операторов Штурма–Лиувилля с сингулярными потенциалами на графах с циклами] // Изв. Сарат. ун-та. Нов. сер. Сер. Математика. Механика. Информатика. 2019. Т. 19, вып. 4. С. 366–376. DOI: <https://doi.org/10.18500/1816-9791-2019-19-4-366-376>