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Bounded finite-time stabilization of the prey – predator model via Korobov's controllability function

A. E. Choque-Rivero✉, **F. Ornelas-Tellez**

Universidad Michoacana de San Nicolas de Hidalgo (UMSNH), Morelia, 58030, Michoacan, Mexico

Abdon E. Choque-Rivero, abdon.choque@umich.mx, https://orcid.org/0000-0003-0226-9612
Fernando Ornelas-Tellez, fornelas@umich.mx, https://orcid.org/0000-0002-6428-8184

Abstract. The problem of finite-time stabilization for a Leslie-Gower prey – predator system through a bounded control input is solved. We use Korobov's controllability function. The trajectory of the resulting motion is ensured for fulfilling a physical restriction that prey and predator cannot achieve negative values. For this purpose, a certain ellipse depending on given data and the equilibrium point of the considered system is constructed. Simulation results show the effectiveness of the proposed control methodology.
Keywords: finite-time stabilization, Korobov's controllability function, bounded control input, prey – predator model
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Стабилизация за конечное время ограниченным управлением модели хищник – жертва с помощью функции управляемости В. И. Коробова

А. Э. Чоке-Риверо✉, **Ф. Орнелас-Тельес**

Мичоаканский университет Сан-Николас-де-Идальго, Мексика, 58030, штат Мичоакан, г. Морелия, университетский городок

Чоке-Риверо Абдон Эдди, кандидат физико-математических наук, ответственный в группе «Уравнения математической физики», abdon.choque@umich.mx, https://orcid.org/0000-0003-0226-9612

Орнелас-Тельес Фернандо, кандидат технических наук, группа «Теория управления», fornelas@umich.mx, https://orcid.org/0000-0002-6428-8184

Аннотация. Рассматривается управляемая модель взаимодействия двух видов жертва – хищник. Численность популяции описывается системой дифференциальных уравнений 2-го порядка, в правую часть которой входит управление, удовлетворяющее наперед заданному ограничению. Система имеет точку покоя (точку равновесия). Необходимо выбрать управление так, чтобы перевести произвольное начальное состояния из некоторой окрестности точки равновесия по траектории системы в точку равновесия за конечное время. Строится семейство позиционных управлений, которое решает эту задачу. Находится окрестность точки покоя, являющаяся эллипсом с центром в этой точке. Причем все траектории, отвечающие этим управлениям и начинающиеся в произвольной точке эллипса, заканчиваются в точке равновесия и находятся внутри эллипса.
Ключевые слова: стабилизация за конечное время, функция управляемости В. И. Коробова, ограниченное управление, модель хищник – жертва

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INTRODUCTION

An interesting system exhibiting oscillations and chaotic behavior is the prey – predator model [1–3], which because of its complex dynamic characteristics results in a challenging system to be controlled [4]. This model has been used to study biological phenomena and the equilibrium of the species. The earliest ratio-dependent model was given by Leslie and Gower [5]. In this model, the predator is also assumed to be growing logistically with a carrying capacity that depends on the availability of a variable resource (prey). This formulation is based on the assumption that a reduction in a predator population has a reciprocal relationship with the per capita availability of its preferred food. This interesting formulation for the predator dynamics has been discussed by Leslie and Gower in [5] and by Pielou in [6].

From a control view point, it is desirable to reach an equilibrium point for the system, particularly in finite time and by a bounded control input, as considered in this work. We will crucially employ V. I. Korobov's method consisting of the use of the controllability function (CF), which is a Lyapunov-type function. The main differences between the CF and the Lyapunov functions are the following.

The use of the CF (resp. Lyapunov funtion) allows stabilizing the control system in finite (resp. in infinite time) [7, 8].

The CF (resp. Lyapunov funtion) is applied to equilibrium or nonequilibrium points (resp. only for equilibrium points [9].

The CF (resp. Lyapunov funtion) is an implicit function (resp. explicit function) [10].

See also [11] and [12].

In comparison with previous works on the stabilization of the prey and predator model [3, 13], in this current work we present a family of bounded controls that stabilize the aforementioned system in finite time. See Theorem 2. Furthermore, for the equilibrium point of the system, an admissible neighborhood in terms of the interior of an ellipse is given. See Lemma 3 and Remark 3.

Notation 1. Let $\mathbb{R}$ denote the set of real numbers. Let S be an $m \times n$ matrix. By $S^{\intercal}$ we denote the transpose matrix of S. Let $x \in \mathbb{R}^n$. By $||x||$ we denote the euclidian norm of $x := (x_1, \ldots, x_n)$, i.e., $||x|| := (x_1^2 + \ldots + x_n^2)^{\frac{1}{2}}$. The norm of an $n \times n$ matrix S is defined by $\|S\| := \max\limits_{1 \leqslant j \leqslant n} \sum_{i=1}^{m} |s_{ij}|$.

1. THE PREY – PREDATOR MODEL

Consider the nonlinear control system

$$\begin{aligned} \dot{x}_1 &= x_1(1 - x_1) - \frac{x_1 x_2}{x_1^2 + \alpha}, \\ \dot{x}_2 &= \gamma \left(1 - \frac{x_2}{\beta x_1}\right) x_2 + u, \quad |u| \leqslant u_1 \end{aligned} \tag{1}$$

defined in set $D := \{(x_1, x_2) \in \mathbb{R}^2 : \ x_1 > 0, \ x_2 \geqslant 0\}$ with the initial condition (x_1^0, x_2^0) and $x_1^0 > 0$, $x_2^0 > 0$, with u being a control input for achieving stabilization of the system at the equilibrium point. Let $(\xi, \eta) \in D$ be the equilibrium point of the system (1) with $u = 0$. In system (1), $x_1 = X/K$, $x_2 = mY/r\,K^2$, $t = r\,T$, $\alpha = a/K^2$, $\beta = m\,n/K\,r$ and $\gamma = s/r$, where X and Y represent the prey and predator population, respectively. The parameter r is the intrinsic growth of prey species with carrying capacity K. Furthermore, T is a scaled time variable, m denotes the per capita consumption rate of the predator. Parameter a denotes the number of prey required to halve the maximum rate, just half, while s is the growth rate of the logistically growing population Y, and finally n is a magnitude of the food quality that the prey provides for conversion into predator population [4, 13]. All the parameters are assumed to be positive.

The statement of the problem we consider is the following: find a *bounded* positional $u = u(x)$ with $|u(x)| \leqslant u_1$ and such that the trajectory $x(t) = (x_1(t), x_2(t))$ starting at the initial point $x_0 := (x_1^0, x_2^0)$ and belonging to a certain neighborhood of the point $\bar{x} := (\xi, \eta)$ terminates at $\bar{x}$ at *finite* time $T(x, \bar{x})$. This problem is called *the synthesis problem*.

System translated to the origin

Recall that the state (x_1, x_2) of the system (1) belongs to the open first quadrant, when $x_1 > 0$ and $x_2 \geqslant 0$. The following remark is valid.

Remark 1. The abscissa and ordinate of the equilibrium point (ξ, η) of the system (1) satisfy the following inequalities:

$$0 < \xi < 1, \quad \text{and} \quad 0 < \eta < \beta.$$

Proof. From the right hand side of (1) with $u = 0$, we have that the abscissa of an equilibrium point belonging to Q_1 satisfies

$$x^3 - x^2 + x(\alpha + \beta) - \alpha = 0. \tag{2}$$

Denote by f the left hand side of (2). Note that the coefficients of f alternate in sign, which implies that (2) has three positive solutions or one positive solution. By analyzing the derivative f', we find that (2) has one positive root and that $0 < \xi < 1$. Consequently, from the second equality of (1) we have that $0 < \eta < \beta$. $\square$

By translating the equilibrium point (ξ, η) to the origin, we have

$$\dot{y} = Ay + bu + g(y), \tag{3}$$

where

$$A := \begin{pmatrix} \frac{2\xi(1-\xi)^2}{\beta} - \xi & -\frac{1-\xi}{\beta} \\ \beta\gamma & -\gamma \end{pmatrix}, \quad b := \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{4}$$

$$g(y) := \begin{pmatrix} g_1(y_1, y_2) \\ g_2(y_2, y_2) \end{pmatrix} \tag{5}$$

$$g_1(y_1, y_2) := -\frac{d_4^2 y_1^4 + d_3 y_1^3 + d_2 y_1^2 + d_1 y_2 y_1}{d_4^2 \left(2y_1\xi + y_1^2 + \alpha + \xi^2\right)}, \quad g_2(y_1, y_2) := -\frac{\gamma \left(y_2 - \beta y_1\right)^2}{\beta \left(\xi + y_1\right)},$$

and

$$d_1 := \left(\alpha - \xi^2\right)\left(\alpha + \xi^2\right), \quad d_2 := \left(\alpha + \xi^2\right)\left(-w\xi + \alpha^2 + \alpha(5\xi - 3)\xi + \xi^3\right),$$
$$d_3 := \left(\alpha + \xi^2\right)\left(3\alpha\xi - \alpha + \xi^3 + \xi^2\right), \quad d_4 := \alpha + \xi^2.$$

We assume that the parameters α and β are positive. Consequently, the function g appearing in (5) and the system (3) is well-defined in the region:

$$D_0 := \{(y_1, y_2) \in \mathbb{R}^2 : \; y_1 + \xi > 0, \; y_2 + \eta > 0\}.$$

The linear part of (3) is completely controllable if and only if $\operatorname{rank}(b, Ab) = 2$, i.e., if and only if

$$\frac{1-\xi}{\beta} \neq 0.$$

In the sequel, we will study the control system in a certain neighborhood of the origin

$$D_1 := \{y \in D_0 : \|g(y)\| \leqslant C_1 \|y\|\} \tag{6}$$

for some $C_1 > 0$.

2. FINITE-TIME STABILIZATION

It seems that [14] was the first to use the term *finite-time stability* (FTS). Further developments in FTS were made by a number of researchers: [15–18] and references therein. See also [19] and [20].

In this work, we employ the theorem appearing in [7, p. 552], where the synthesis of bounded controls in the first approximation of a certain general nonlinear system is treated. Our work differs from [7] mainly because we construct a specific control that depends on the equilibrium point $\bar{x}$, which in turn depends on the parameters of the system α, β and γ. We also describe a certain ellipse “centered” at the equilibrium point $\bar{x}$, so from every inner point x_0 of this ellipse it is possible to arrive at $\bar{x}$ in finite time $T(x, \bar{x})$. Another important novelty is the fact that in the construction of the bounded control $u(x)$ we use the method proposed in [21]. See also [22] and [23]. The controls appearing in [21] depend on a parameter (as in (12)) that in turn enables having a family of controls that could solve the synthesis problem.

By Remark 1, $\xi - 1 \neq 0$. Let

$$F := \begin{pmatrix} \frac{\beta}{\xi-1} & 0 \\ \frac{\left(2(\xi-1)^2-\beta\right)\xi}{\xi-1} & 1 \end{pmatrix}. \tag{7}$$

Clearly, we see that $\det F \neq 0$.

Remark 2. The matrix F can be written as $F = \begin{pmatrix} c^\intercal \\ c^\intercal A \end{pmatrix}$, where c is a vector satisfying $(c, b) = 0$ and $(c, Ab) = 1$.

Furthermore, we use the transformation

$$z = Fy \tag{8}$$

to rewrite Eq. (3) in the canonical form

$$\dot{z} = A_0 z + bw + Fg(F^{-1}z) \quad |w| \leqslant w_1, \tag{9}$$

where

$$p = (p_1, p_2)^\intercal \tag{10}$$

with $p_1 := -\frac{\gamma\left(\eta - 2(\xi-1)^2\xi^2\right)}{\eta}$ and $p_2 := \frac{\xi\left(2(\xi-1)^2\xi-\eta\right)}{\eta} - \gamma$ and $A_0 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. The new control w has the following form:

$$w := p^\intercal z + u \tag{11}$$

with the restriction $w \leqslant w_1$, where

$$w_1 := u_1 - u_2 \sum_{j=1}^{2} |p_j|. \tag{12}$$

We assume that $u_2 < \frac{u_1}{\sum_{j=1}^{2} |p_j|}$. As in [7], we require that system (9) is considered in the neighborhood

$$Q := \{z : |z_j| \leqslant u_2\}. \tag{13}$$

Our next step is to construct a positional control $w(z)$ such that $|w| \leqslant w_1$ and that the trajectory of any initial point $z_0 := (z_1^0, z_2^0)$ belonging to a certain neighborhood of the origin arrives to the origin at finite time $T(z_0)$. To this end, we will use V. I. Korobov's method, which consists of a Lyapunov-type function $\theta(z)$, that is the only positive solution of the following equation:

$$2a_0\theta = (K(\theta)z, z), \tag{14}$$

where

$$K(\theta) := \frac{1}{4 + a_1} \begin{pmatrix} \frac{a_1}{\theta^3} & -\frac{2}{\theta^2} \\ -\frac{2}{\theta^2} & -\frac{1}{\theta} \end{pmatrix}$$

is a positive matrix for $\theta > 0$. The number a_1 is a negative number such that the matrices K and $\frac{1}{\theta}K - \frac{d}{d\theta}K$ are both positive definite matrices. In terms of the parameter a_1, this condition is equivalent to the following inequality:

$$a_1 < -\frac{9}{2}. \tag{15}$$

The number a_0 satisfies the inequality

$$a_0 \leqslant \frac{w_1^2}{2a_1(a_1+3)}. \tag{16}$$

In the frame of Korobov's method, the positional control $w(z)$ has the form

$$w(z) := \frac{a_1 z_1}{\theta^2(z_1,z_2)} - \frac{3z_2}{\theta(z_1,z_2)}. \tag{17}$$

Recall that in [21] for the linear system $\dot{z} = A_0 z + bw$, a family of bounded positional controls was proposed that exactly stabilized this system at time $T(z_0) = \theta_0$, where θ_0 is the root of Eq. (14) for z_0.

Let us now rewrite the matrices K and $\frac{1}{\theta}K - \frac{d}{d\theta}K$ in a more convenient form.

Let $D(\theta) := \begin{pmatrix} \theta^{-\frac{3}{2}} & 0 \\ 0 & \theta^{-\frac{1}{2}} \end{pmatrix}$. Thus, the matrices $K = K(\theta)$ and $\frac{1}{\theta}K - \frac{d}{d\theta}K$ can be written as follows:

$$D(\theta)K_1D(\theta) = K, \quad \frac{1}{\theta}D(\theta)K_2D(\theta) = \frac{1}{\theta}K - \frac{d}{d\theta}K,$$

where

$$K_1 := \frac{1}{4+a_1}\begin{pmatrix} a_1 & -2 \\ -2 & -1 \end{pmatrix}, \quad K_2 := \frac{1}{4+a_1}\begin{pmatrix} 4a_1 & -6 \\ -6 & -2 \end{pmatrix}. \tag{18}$$

In the sequel, we assume that θ satisfies the inequality $\theta \leqslant 1$.

Lemma 1. *Let $\lambda_{\min,K_2}$ be the minimal eigenvalue of the matrix K_2 and C_1 the constant appearing in (6). Thus, the following is valid:*

$$\frac{(Kz, Fg(F^{-1}z))}{((\frac{1}{\theta}K - \frac{d}{d\theta}K)z, z)} \leqslant \theta\frac{C_1\|K_1\|}{\lambda_{\min,K_2}}.$$

Proof. Denote

$$q := D(\theta)z. \tag{19}$$

By using (18) and (19), we then have

$$\frac{(Kz, Fg(F^{-1}z))}{((\frac{1}{\theta}K - \frac{d}{d\theta}K)z, z)} = \frac{(DK_1Dz, Fg(F^{-1}z))}{\frac{1}{\theta}(DK_2Dz, z)} = \frac{(K_1q, DFg(F^{-1}D^{-1}q))}{\frac{1}{\theta}(K_2q, q)} \leqslant$$
$$\leqslant \theta C_1\frac{\|K_1\|\|q\|\|D\|\|F\|\|F^{-1}\|\|D^{-1}\|\|q\|}{\lambda_{\min,K_2}\|q\|^2} = \theta\frac{C_1\|K_1\|}{\lambda_{\min,K_2}}.$$

□

The following result gives an estimation of the derivative of the controllability function θ with respect to the system (9).

Theorem 1. *Let C_1, K_1 and $\lambda_{\min,K_2}$ be as in (6), (18) and the minimal eigenvalues of K_2, respectively. The following inequality is valid:*

$$\dot{\theta} \leqslant -1 + \theta \frac{C_1 \|K_1\|}{\lambda_{\min,K_2}}. \tag{20}$$

Proof. Let $a := \left(\frac{a_1}{\theta^2}, \frac{-3}{\theta}\right)^{\intercal}$. We take the derivative of Eq. (14) with respect to system (9), and we have

$$\dot{\theta} = \frac{((KA_0 + A_0^{\intercal}K + ab^{\intercal}K + Kba^{\intercal})z, z)}{((\frac{1}{\theta}K - \frac{d}{d\theta}K)z, z)} + 2\frac{(Kz, Fg(F^{-1}z))}{((\frac{1}{\theta}K - \frac{d}{d\theta}K)z, z)}, \tag{21}$$

$$= -1 + 2\frac{(Kz, Fg(F^{-1}z))}{((\frac{1}{\theta}K - \frac{d}{d\theta}K)z, z)}. \tag{22}$$

The first term of the right side of (21) is equal to -1 because of [21, Equation (2.9)]. Finally, inequality (20) readily follows from (22) and Lemma 1. □

Notation 2. Let $\widehat{\theta} > 0$, $C_2 > 0$ such that for $\theta \leqslant \widehat{\theta}$

$$-1 + \theta \frac{C_1 \|K_1\|}{\lambda_{\min,K_2}} \leqslant -C_2. \tag{23}$$

Let

$$D_2 := \{z : 0 < \theta(z) < \widehat{\theta}\}. \tag{24}$$

Lemma 2. *Let inequality (23) holds. Thus, the following inequality is valid:*

$$\dot{\theta} \leqslant -C_2. \tag{25}$$

Moreover, for $z \in D_2$, the time in reaching from z to the origin is estimated by the following inequality:

$$T(z) \leqslant \frac{\theta_0}{C_2}. \tag{26}$$

Proof. By taking into account (20) and (23), inequality (25) readily follows. To prove (26), one integrates (25) on the trajectory $z = z(t)$ and attains $\theta(z(t)) - \theta_0 \leqslant -C_2 t$. By using [7, p. 552], we have that $z(T) = 0$, which implies $\theta(z(T)) = 0$. Thus, we obtain (26). □

Next, to guarantee that the trajectory $x(t)$ starting near the equilibrium point (ξ, η) will not abandon the first quadrant, we rewrite equality (14) in a more convenient form:

$$2a_0\theta^4 = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}^{\intercal} \begin{pmatrix} \frac{a_1}{4+a_1} & -\frac{2\theta}{4+a_1} \\ -\frac{2\theta}{4+a_1} & -\frac{\theta^2}{4+a_1} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}. \tag{27}$$

In terms of the variable $y = (y_1, y_2)^{\intercal}$, equality (27) is written as $2a_0\theta^4 = (K_3(\theta)y, y)$, where

$$K_3(\theta) := F^{\intercal} \begin{pmatrix} \frac{a_1}{4+a_1} & -\frac{2\theta}{4+a_1} \\ -\frac{2\theta}{4+a_1} & -\frac{\theta^2}{4+a_1} \end{pmatrix} F$$

and F as in (7). Let $\lambda^1_{K_3}(\theta)$ and $\lambda^2_{K_3}(\theta)$ be the eigenvalues of $K_3(\theta)$. Note that for fixed positive θ these eigenvalues are positive numbers. Recall from (15) that $a_1 < -\frac{9}{2}$.

The following lemma can be verified by using facts concerning the transformation of the quadratic to a canonical form [24].

Lemma 3. *With $\xi > 0, \eta > 0$, let $\bar{x} = (\xi, \eta)$ be an equilibrium point of the system (1), and let K_1 be as in (18). Furthermore, let $(k_{j,\ell})_{j,\ell=1}^2 := K_1$, and the parameter a_0 satisfies (16). Let $\widetilde{\theta}$ be the smallest positive value for which the following inequality holds:*

$$\max\left\{\frac{2a_0\theta^4}{\lambda^1_{K_3}(\theta)}, \frac{2a_0\theta^4}{\lambda^2_{K_3}(\theta)}\right\} \leqslant \min\{\xi, \eta\}.$$

Thus, the interior of the ellipse

$$D_{\mathscr{E}} := \mathscr{E}(x, \widetilde{\theta}, \bar{x}) = 0 \tag{28}$$

belongs to the first quadrant. Here

$$\mathscr{E}(x, \theta, \bar{x}) := 2a_0\theta^4 - \sum_{j,\ell=1}^{2} k_{j,\ell}\theta^{j+\ell-2}(c, A^{j-1}(x-\bar{x}))(c, A^{\ell-1}(x-\bar{x})). \tag{29}$$

Notation 3. Let Q, D_2 and $D_{\mathscr{E}}$ be as in (13), (24) and (28), respectively.
Let

$$D_3 := Q \cap D_2 \cap D_{\mathscr{E}}. \tag{30}$$

Here Q and D_2 are understood in terms of the variable x.

Now we present the main result of our work.

Theorem 2. *Let $\bar{x} = (\xi, \eta)$ and $D_{\mathscr{E}}$ be as in Lemma 3. Let A be as in (4), c be as in Remark 2 and $a_1 < -\frac{9}{2}$, $a_2 = -3$. Furthermore, let p_j for $j = 1, 2$ be as in (10) and suppose that (x_1^0, x_2^0) belongs to the region D_3. Thus, the control*

$$u(x, \bar{x}) = \sum_{j=1}^{2} a_j\theta^{j-3}(x-\bar{x})(c, A^{j-1}(x-\bar{x})) - \sum_{j=1}^{2} p_j(x-\bar{x})(c, A^{j-1}(x-\bar{x})) \tag{31}$$

satisfies the condition $|u(x)| \leqslant u_1$ and solves the synthesis problem. The time of motion from $x_0 = (x_1^0, x_2^0)$ to the origin satisfies the following inequality

$$T(x_0, \bar{x}) \leqslant \frac{\theta_0}{C_2}. \tag{32}$$

Proof. The restriction $|u(x)| \leqslant u_1$ is verified by employing (11), (12) and the inequality $|w| \leqslant w_1$. This last inequality is proved in [21, Theorem 3.1]. Inequality (32) follows from Theorem 1 and Lemma 2. □

Remark 3. For fixed θ, the equation $\mathscr{E}(x, \theta, \bar{x}) = 0$ represents an ellipse. Since the control (31) stabilizes the system (1), the trajectory of the system (1) will not leave the ellipse (28) calculated at $\theta = \theta_0$. In turn, θ_0 is the solution of the equation $\mathscr{E}(x, \theta, \bar{x}) = 0$ for $x = x_0$.

Remark 4. We emphasize that the trajectory $x(t)$ under the influence of control $u(x, \bar{x})$ approaches the equilibrium point $\bar{x}$ for $t \to T = T(x_0, \bar{x})$. For $t > T$, the trajectory stays at the equilibrium point $\bar{x}$.

3. GRAPH OF THE TRAJECTORY AND CONTROL

To plot the graph of the trajectory $x(t)$ from a given initial point (x_1^0, x_2^0), as well as the control $u(x(t))$ and the controllability function $\theta(x(t))$, we have added a differential equation for the variable θ:

$$\begin{aligned} \dot{x}_1 &= x_1(1-x_1) - \frac{x_1 x_2}{x_1^2+\alpha}, \\ \dot{x}_2 &= \gamma\left(1 - \frac{x_2}{\beta x_1} x_2\right) + u(x_1, x_2), \\ \dot{\theta} &= -1 + 2\psi(x, \theta, \bar{x}), \end{aligned} \tag{33}$$

with initial conditions $x_1(0) = x_1^0$, $x_2(0) = x_2^0$ and $\theta(0) = \theta_0$. Here θ_0 is the root of (29). Moreover,

$$\psi(x, \theta, \bar{x}) := \frac{(D(\theta)K_1 D(\theta)(x-\bar{x}), \quad Fg(F^{-1}(x-\bar{x})))}{\frac{1}{\theta}(D(\theta)K_2 D(\theta)(x-\bar{x}), (x-\bar{x}))}.$$

Let us remark that the initial point (x_1^0, x_2^0) should belong to the region D_3 (30).

Example

Let $\alpha = 10$, $\beta = \frac{1081}{900}$, $\gamma = 1$ and $u_1 = 5$. The equilibrium point $\bar{x} = (\xi, \eta)$ is equal to $\left(\frac{9}{10}, \frac{1081}{1000}\right)$. The vector p is equal to $\left(-\frac{5324}{5405}, -\frac{20377}{10810}\right)$, $w_1 = 1$, $a_1 = -6$ and by (12), $2a_0 = \frac{1}{18}$. The positional control has the form

$$u(x, \bar{x}) = -\frac{\frac{3243}{50} - \frac{1081 x_1}{15}}{\theta^2} - \frac{\frac{3189 x_1}{100} + 3x_2 - \frac{3993}{125}}{\theta} + \frac{79841879 x_1}{9729000} + \frac{31187 x_2}{10810} - \frac{14084927}{1351250}$$

with $\theta = \theta(x - \bar{x})$.

The function $\psi(x, \theta, \bar{x})$ is given by $\psi(x, \theta, \bar{x}) = \dfrac{\mathrm{num}(x, \bar{x})}{\mathrm{den}(x, \bar{x})}$, where

$$\begin{gathered} \mathrm{num}(x, \bar{x}) := x_3 \left(648600 y_1 \left(10810 y_1^3 + 18539 y_1^2 + (114229 - 900 y_2) y_1 + 9190 y_2\right) x_1 \times \right. \\ \times (y_1 (3189 x_3 - 10810) + 300 y_2 x_3) - \left(1034192700 y_1^5 + 2821255660 y_1^4 - \right. \\ -9 (31187000 y_2 - 1414988103) y_1^3 + 4 \left(20250000 y_2^2 + 112867650 y_2 + 2774668507\right) y_1^2 + \\ \left.\left. +90 y_2 (1620000 y_2 - 14579147) y_1 + 875610000 y_2^2\right) x_3 (y_1 (9567 x_3 - 21620) + 900 y_2 x_3)\right), \\ \mathrm{den}(x, \bar{x}) := \frac{3243}{50} \left(100 y_1^2 + 180 y_1 + 1081\right) x_1 \left(46742440000 y_1^2 - 1945800 y_1 x_3 (10630 y_1 + \right. \\ \left. +1000 x_2 - 1081) + 27 x_3^2 (10630 y_1 + 1000 x_2 - 1081)^{\,2}\right) \end{gathered}$$

with $y_1 = x_1 - \frac{9}{10}$ and $y_2 = x_2 - \frac{1081}{1000}$.

With the initial conditions $x_1^0 = 1$ and $x_2^0 = \frac{1}{2}$, the graph in Fig. 1 shows the trajectories of $x_1(t)$ and $x_2(t)$, while in Fig. 2 the system phase portrait is displayed.

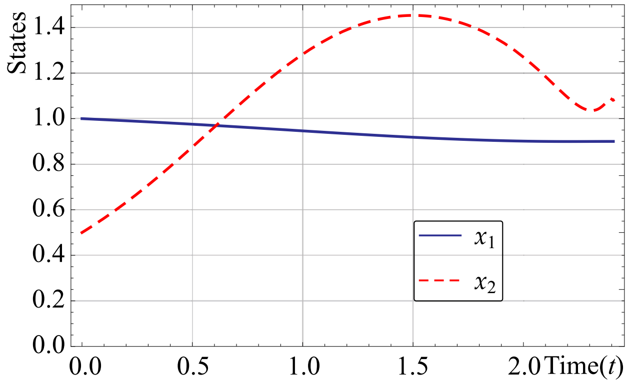

Fig. 1. Trajectories of $x_1(t)$ and $x_2(t)$

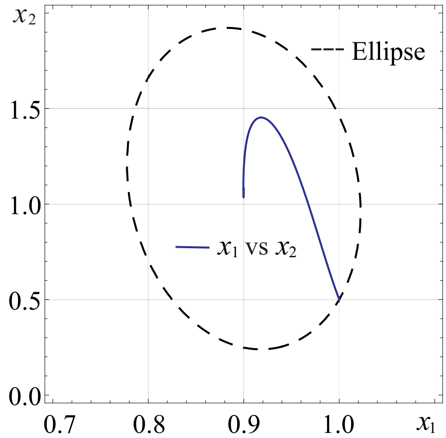

Fig. 2. System phase portrait $x_1(t)$ vs $x_2(t)$ and the corresponding ellipse

Fig. 3 shows the controllability function θ on the trajectory $x(t)$. The graph of the positional control u on the trajectory $x(t)$ is as shown in Fig. 4.

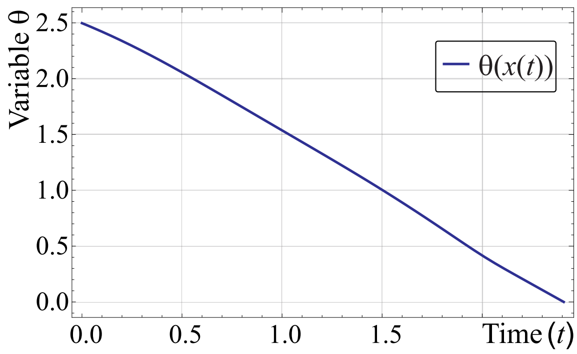

Fig. 3. The controllability function time evolution

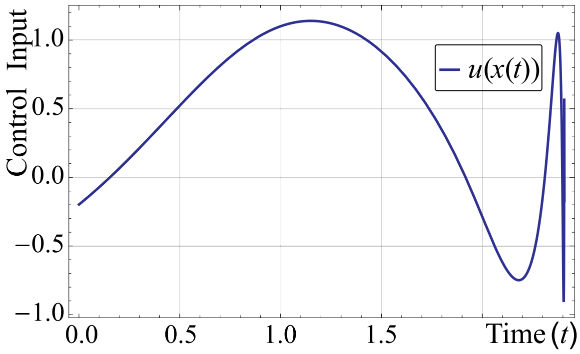

Fig. 4. The positional control input

By using Wolfram Mathematica, we have calculated that the time of arriving from x_0 to $\bar{x}$ is $T(x_0, \bar{x}) = 2.407427$ and that $x_1(T) = \xi$ and $|x_2(T) - \eta| \leqslant 1.06063 \times 10^{-9}$.

The corresponding ellipse for the equilibrium point $\left(\frac{9}{10}, \frac{1081}{1000}\right)$ is given by the equality

$$\frac{\theta^4}{18} - \frac{1}{2}\theta^2 y_2^2 + \left(-\frac{1129969\theta^2}{20000} + \frac{1149103\theta}{4500} - \frac{1168561}{2700}\right) y_1^2 + \left(\frac{1081\theta}{45} - \frac{1063\theta^2}{100}\right) y_1 y_2 = 0$$

for $\theta = 2.497329$ and $y_1 = x_1 - \frac{9}{10}$ and $y_2 = x_2 - \frac{1081}{1000}$.

To the best of the authors' knowledge, no control methodologies have been applied to this system, which considers two main features: achieving finite-time convergence with a bounded control input.

CONCLUSIONS

We have presented a family of explicit bounded controls that stabilize the predator – prey system (1) in finite time. An ellipse depending on the parameters of the system (1) and the equilibrium point is given. The translation of any initial point $x_0 = (x_1^0, x_2^0)$ to the equilibrium point $\bar{x}$ is guaranteed if x_0 belongs to this ellipse, which in turn is located in the first quadrant of $\mathbb{R}^2$: the initial point satisfies the conditions $x_1^0 > 0$ and $x_2^0 \geqslant 0$.

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