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Numerical solution of linear differential equations with discontinuous coefficients and Henstock integral

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Abstract. We consider the problem of approximate solution of linear differential equations with discontinuous coefficients. We assume that these coefficients have f-primitive. It means that these coefficients are Henstock integrable only. Instead of the original Cauchy problem, we consider a different problem with piecewise-constant coefficients. The sharp solution of this new problem is the approximate solution of the original Cauchy problem. We found the degree of approximation in terms of f-primitive for Henstock integrable coefficients. Two examples are given. In the first example, the coefficients have an infinite derivative at zero. In the second example, the coefficients have an infinite derivative at interior points.
Keywords: linear differential equations, Cauchy problem, Henstock integral, numerical solution
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Численное решение линейных дифференциальных уравнений с разрывными коэффициентами и интеграл Хенстока

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Аннотация. Рассматривается задача приближенного решения линейных дифференциальных уравнений с разрывными коэффициентами. Предполагается, что эти коэффициенты имеют f-примитивные. Это означает, что эти коэффициенты являются интегрируемыми только по Хенстоку. Вместо исходной задачи Коши мы рассматриваем другую задачу с кусочно-постоянными коэффициентами. Точное решение этой новой задачи есть приближенное решение исходной задачи Коши. Мы указываем степень аппроксимации в терминах f-примитивных для интегрируемых по Хенстоку коэффициентов. Приведены два примера. В первом примере коэффициенты имеют бесконечную производную в нуле. Во втором примере коэффициенты имеют бесконечную производную во внутренних точках.

Ключевые слова: линейные дифференциальные уравнения, задача Коши, интеграл Хенстока, численное решение

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Introduction

In the classical initial value problem for a linear differential equation of the first order

$$y\prime + p(x)y = q(x), \quad y(a) = y_0, \quad x \in [a, b], \tag{1}$$

the coefficients $p(x)$ and $q(x)$ are continuous functions. However, some problems of dry friction and electric circuit with relay are given the equations with the discontinuous functions p and q. For example, the RL-electric circuit with relay is described by a linear differential equation

$$\frac{di}{dt} + \frac{R(t)}{L} i = \frac{e(t)}{L}$$

with the discontinuous function $R(t)$. In this case, it is assumed that the functions $p(x)$ and $p(x)$ are (L)-integrable and a function $y(x)$ is called a solution to equation (1) if $y(x)$ is absolutely continuous and satisfies the equation (1) almost everywhere on $[a, b]$.

There are no effective methods for the approximate solution of equations with unbounded coefficients $p(x)$ and $q(x)$. If the coefficients $p(x)$ and $q(x)$ are unbounded in some neighborhood of the point a, then Runge – Kutta method does not work. If the coefficients $p(x)$ and $q(x)$ are unbounded in some neighborhood of the interior point $c \in (a, b)$, then Runge – Kutta method has a very large error, usually more than 1.

Some authors use Haar and Walsh functions to solve linear equations [1–3]. In [4,5] G. Gat and R. Toledo propose to approach the solution $y(x)$ by the Walsh polynomial

$$\tilde{y}_n(x) = \sum_{k=0}^{2^n-1} c_k W_k(x).$$

In [4] for continuous function $q(x)$ $(x \in [0,1])$ and $p(x) = \text{const}$, an estimate for the error $|y(x) - \tilde{y}(x)|$ is obtained. In [5] the authors consider the case when $q \in L(0,1)$ is a continuous function on $[0,1[$ and prove that $\tilde{y}_n(x)$ converges uniformly to the solution $y(x)$ on the interval $[0,1[$.

In [6], the authors present the derivative y' of the solution y as a Haar expansion and obtain an estimate of the approximate solution in terms of the modulus of continuity of the coefficients $p(x)$ and $q(x)$. This method can also be used for equations with unbounded coefficients $p(x)$ and $q(x)$.

In this article we will assume that $p(x)$ and $q(x)$ are Henstock integrable functions on the interval $[a,b]$. We construct the approximate solution $\tilde{y}(x)$ and obtain the estimate of the error $|y(x)-\tilde{y}(x)|$ in terms of modulus of continuity $\omega_{\frac{1}{2^n}}(e^P)$, $\omega_{\frac{1}{2^n}}(e^{-P})$, and $\omega_{\frac{1}{2^n}}(Q)$, where P and Q are f-primitives for p and q respectively.

The paper is organized as follows. In Sec. 1, we recall some facts from Henstock integral. In Sec. 2, we indicate the necessary and sufficient condition, under which the Cauchy problem has a solution. This solution is given in terms of the Henstock integral. In Sec. 3, we construct the approximative solution and find the error. In Sec. 4, we give two examples.

1. Henstock integral on the interval

Any function $\delta(x) > 0$ on $[a,b]$ is said to be a gauge. Let $\mathfrak{X} = (x_k)_{k=0}^n$ be a partition of the interval $[a,b]$. The point $\xi_k \in [x_{k-1}, x_k]$ is called a tag of $[x_{k-1}, x_k]$, the set of ordered pairs $([x_{k-1}, x_k], \xi_k)_{k=1}^n$ is called a tagged partition and is denoted by $\overset{\circ}{\mathfrak{X}} = ([x_{k-1}, x_k], \xi_k)_{k=1}^n$.

The tagged partition $\overset{\circ}{\mathfrak{X}} = ([x_{k-1}, x_k], \xi_k)_{k=1}^n$ of the interval $[a,b]$ is called δ-fine and is denoted by $\overset{\circ}{\mathfrak{X}} \ll \delta$ if for any $k = 1, \dots, n$

$$|x_{k-1} - x_k| < \delta(\xi_k).$$

It is known that for any gauge $\delta(x) > 0$ on $[a,b]$ there exists a δ-fine partition $\overset{\circ}{\mathfrak{X}} = ([x_{k-1}, x_k], \xi_k)_{k=1}^n$ of $[a,b]$.

A function $f : [a,b] \to \mathbb{R}$ is said to be Henstock-integrable (or generalized Riemann-integrable) on the interval $[a,b]$ if there exists a number $I(f) \in \mathbb{R}$ such that

$$\forall\, \varepsilon > 0\ \exists\ \delta(x) > 0 \text{ on } [a,b]\ \forall\ \overset{\circ}{\mathfrak{X}} \ll \delta(x), \quad |S(\overset{\circ}{\mathfrak{X}}, f) - I(f)| < \varepsilon.$$

The number $I(f)$ is called Henstock integral and is denoted by $(R^*)\int\limits_a^b f(x)\,dx$ or $\int\limits_a^b f(x)\,dx$.

The collection of all functions that are Henstock integrable on $[a,b]$ is denoted by $R^*(a,b)$. A function $f : [a,b] \to \mathbb{R}$ is called absolutely integrable if $f \in R^*(a,b)$ and $|f| \in R^*(a,b)$. There exist Henstock integrable functions that are not absolutely integrable. If the function $f : [a,b] \to \mathbb{R}$ is absolutely integrable, then $f \in L(a,b)$.

The function $G : [a,b] \to R$ is called a c-primitive (f-primitive) for a function g if G is continuous on $[a,b]$ and there exists a countable (finite) set $E \subset [a,b]$ such that $G'(x) = g(x)$ on $[a,b] \backslash E$. We will use the following properties of Henstock integral.

Theorem 1 ([7]). *If $f : [a,b] \to R$ has a c-primitive F with a exceptional set E, then $f \in R^*(a,b)$ and for all x*

$$\int_a^x f(t)dt = F(x) - F(a).$$

It follows that for $x \in [a,b] \setminus E$

$$\frac{d}{dx}\int_a^x f(t)dt = f(x).$$

Theorem 2 ([7]). *Let $f \in R^*(a,b)$ and $F(x) = \int_a^x f(t)dt$. The function f is absolutely integrable on $[a,b]$ if and only if $\bigvee_a^b(F) < +\infty$. In this case,*

$$\int_a^b |f(t)|dt = \bigvee_a^b(F).$$

Theorem 3 ([7]). *If $f \in R^*(a,b)$ and g is monotone on $[a,b]$, then there exists $\xi \in [a,b]$ such that*

$$\int_a^b f(x)g(x)dx = g(a)\int_a^\xi f(x)dx + g(b)\int_\xi^b f(x)dx.$$

Theorem 4 ([7]). *Let F and G be a c-primitives on $[a,b]$. Then $F'G \in R^*(a,b)$ if and only if $FG' \in R^*(a,b)$. In this case,*

$$\int_a^b F'Gdt = F(t)G(t)\Big|_a^b - \int_a^b FG'dt.$$

Theorem 5 ([7], Hake's theorem). *Let $f : [a,b] \to \mathbb{R}$ and $f \in R^*(a,c)$ for any $c \in (a,b)$. Then $f \in R^*(a,b)$ if and only if there exists*

$$\lim_{c \to b-0}\int_a^c f(x)\,dx = I.$$

In this case, $I = (R^*)\int_a^b f(x)\,dx$.

A detailed exposition of the Henstock integral theory can be found in [7, 8].

2. Linear differential equations and Henstock integral

Let $p, q : [a,b] \to \mathbb{R}$ be two continuous functions that the differentiable on the interval $[a,b]$ with the exception of a countable set E. We will consider the classical Cauchy initial value problem

$$y' + p'(x)y = q'(x), \quad x \in [a,b] \setminus E, \tag{2}$$

$$y(a) = y_0. \tag{3}$$

It follows from Theorem 1 that functions functions $p'(x)$ and $q'(x)$ are Henstock integrable. This is a weaker condition than $p', q' \in L(a,b)$.

Example 1. Define the function q on $[a, b]$ in the following way. Let $x_n = a + \frac{b-a}{2^n}$. Assume $q(a) = q(x_n) = 0, q(\frac{x_n+x_{n+1}}{2}) = \frac{1}{n}$, $q(x)$ is lineal on $[x_{n+1}, \frac{x_n+x_{n+1}}{2}]$ and $[\frac{x_n+x_{n+1}}{2}, x_n]$. Then $q' \in R^*(a, b)$, but $q' \notin L(a, b)$.

Theorem 6. *Equation (2) has a continuous solution that is differentiable on the set $[a, b] \setminus E$ if and only if the function $e^{p(x)}q'(x)$ has a c-primitive differentiable on $[a, b] \setminus E$.*

Proof. *Necessity*. Let $y(x)$ be a solution of (2), that is

$$y'(x) + p'(x)y(x) = q'(x)$$

for all $x \in [a, b] \setminus E$. Then

$$e^{p(x)}y'(x) + p'(x)y(x)e^{p(x)} = q'(x)e^{p(x)}$$

for all $x \in [a, b] \setminus E$ or in another words

$$(y(x)e^{p(x)})' = q'(x)e^{p(x)} \qquad (x \in [a, b] \setminus E). \tag{4}$$

It means that the function $q'(x)e^{p(x)}$ has c-primitive $y(x)e^{p(x)}$.

Sufficiently. Let $q'(x)e^{p(x)}$ has c-primitibe $F(x)$, that is

$$F'(x) = q'(x)e^{p(x)} \quad x \in [a, b] \setminus E.$$

Let us denote $y(x) = \frac{F(x)}{e^{p(x)}} \Leftrightarrow F(x) = y(x)e^{p(x)}$ $(x \in [a, b] \setminus E)$. Then

$$\forall\ x \in [a, b] \setminus E \quad y'(x)e^{p(x)} + y(x)e^{p(x)}p'(x) = q'(x)e^{p(x)} \Leftrightarrow y'(x) + y(x)p'(x) = q'(x).$$ □

Corollary. *A solution of Cauchy initial value problem (2)–(3) is given by the formula*

$$y(x) = e^{p(a)-p(x)}g(a) + e^{-p(x)} \int_a^x q'(t)e^{p(t)}\, dt,$$

where an integral is the Henstock integral.

Proof. It follows from Equality (4) that the function $y(x)e^{p(x)}$ is c-primitive for $q'e^{p(x)}$, it means $q'e^{p(x)}$ is Henstock integrable and the equality

$$\int_a^x q'(t)e^{p(t)}\, dt = y(x)e^{p(x)} - y(a)e^{p(a)}$$

holds. □

Example 2. It is possible to construct the continuous functions p and q so that the function $q'(x)e^{p(x)}$ has a c-primitive, but $q'(x)e^{p(x)} \notin L(a, b)$. For simplicity, we consider the case $[a, b] = [0, 1]$ and select the function $q(x)$ as in Example 1. In this case $x_n = 2^{-n}$, $q(x_n) = q(0) = 0$, $q(x)$ is lineal on $[x_{n+1}, \xi_n]$ and $[\xi_n, x_n]$, where $\xi_n = \frac{1}{2}(x_x + x_{n+1})$. Now we define the function $p(x)$ by the conditions:

(a) $e^{p(2^{-n})} = \beta_n > 1, \beta_n \downarrow 1$ $(n \to \infty)$;

(b) $e^{p(x)}$ is lineal on any interval $[2^{-n-1}, 2^{-n}]$.

It is evident that the series

$$\sum_{n=1}^{\infty} \int_{2^{-k+1}}^{2^{-k}} q'(x)e^{p(x)}\,dx$$

converges. It follows from the Hake theorem that $f(x) = q'(x)e^{p(x)} \in R^*(0,1)$. Therefore the function $F(x) = \int_0^x f(t)dt$ is continuous. Since the function $f(x) = q'(x)e^{p(x)}$ is continuous on any interval $(2^{-n-1}, 2^{-n})$, it follows that $F'(x) = q'(x)e^{p(x)}$ on any interval $(2^{-n-1}, 2^{-n})$. It means that $F(x)$ is c-primitive for $q'(x)e^{p(x)}$. It is not difficult to check that $f(x) = q'(x)e^{p(x)} \notin L(0,1)$.

3. Approximate solution of Cauchy problem (2)–(3) on interval [0,1]

Now we will find an approximate solution of Cauchy initial value problem

$$y' + p'(x)y = q'(x), \quad x \in [0,1] \setminus E, \tag{5}$$

$$y(0) = y_0. \tag{6}$$

We assume that the functions p and q are continuous and have derivatives with the exception of some countable set E. We also assume that $e^{p(x)}q'(x)$ has a c-primitive differentiable on $[a,b] \setminus E$.

We choose an arbitrary $n \in \mathbb{N}$, define the functions $\tilde{p}(x)$ and $\tilde{q}(x)$ by equalities

$$\tilde{p}\left(\frac{k}{2^n}\right) = p\left(\frac{k}{2^n}\right), \quad \tilde{q}\left(\frac{k}{2^n}\right) = q\left(\frac{k}{2^n}\right),$$

$$\tilde{p}(x) = p\left(\frac{k}{2^n}\right) + 2^n\left(x - \frac{k}{2^n}\right)\left(p\left(\frac{k+1}{2^n}\right) - p\left(\frac{k}{2^n}\right)\right), \quad x \in \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right],$$

$$\tilde{q}(x) = q\left(\frac{k}{2^n}\right) + 2^n\left(x - \frac{k}{2^n}\right)\left(q\left(\frac{k+1}{2^n}\right) - q\left(\frac{k}{2^n}\right)\right), \quad x \in \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right],$$

and consider the Cauchy initial value problem

$$\tilde{y}' + \tilde{p}'\tilde{y} = \tilde{q}', \tag{7}$$

$$\tilde{y}(0) = y_0. \tag{8}$$

It is evident that the function $e^{\tilde{p}(x)}\tilde{q}'$ has a f-primitive. By Theorem 6 the functions

$$y(x) = y_0 e^{p(0)-p(x)} + e^{-p(x)} \int_0^x q'(t)e^{p(t)}\,dt,$$

$$\tilde{y}(x) = y_0 e^{\tilde{p}(0)-\tilde{p}(x)} + e^{-\tilde{p}(x)} \int_0^x \tilde{q}'(t)e^{\tilde{p}(t)}\,dt$$

are solutions of Cauchy problems (5)–(6) and (7)–(8) respectively. The function $\tilde{y}(x)$ is the approximate solution of Cauchy problem (5)–(6). In the following theorem, we indicate an estimate for the distance $y(x) - \tilde{y}(x)$.

Theorem 7. *The following inequality*

$$|y(x)-\tilde{y}(x)| \leqslant C_{-1}\omega_{\frac{1}{2^n}}(e^{-p}) + C_1\omega_{\frac{1}{2^n}}(e^{p}) + \omega_{\frac{1}{2^n}}(q)C_0 + C_2\omega_{\frac{1}{2^n}}(q)\omega_{\frac{1}{2^n}}(p),$$

holds, where

$$C_{-1} = |y_0|e^{p(0)} + \|e^p\|_{C[0,1]}\bigvee_0^1 q, \quad C_1 = 2\|e^p\|_{C[0,1]}\bigvee_0^1 q,$$

$$C_0 = \|e^p\|_{C[0,1]} + \|e^p\|_{C[0,1]}\bigvee_0^1 e^p, \quad C_2 = \|e^p\|^2_{C[0,1]}.$$

Proof. 1. First we estimate the difference $y(x)-\tilde{y}(x)$ for $x=\frac{k}{2^n}, k=0,1,...,2^n$. We have

$$y(x)-\tilde{y}(x) = e^{-p(\frac{k}{2^n})}\int\limits_0^{\frac{k}{2^n}} (q'(t)e^{p(t)} - \tilde{q}'(t)e^{\tilde{p}(t)})\,dt =$$

$$= e^{-p(\frac{k}{2^n})}\int_0^{\frac{k}{2^n}} (q'(t)-\tilde{q}'(t))e^{p(t)}\,dt + e^{-p(\frac{k}{2^n})}\int\limits_0^{\frac{k}{2^n}} \tilde{q}'(t)(e^{p(t)} - e^{\tilde{p}(t)})\,dt = I_1 + I_2.$$

To estimate integrals in I_1 and I_2 we will assume that p' and q' — are Henstock absolutely integrable.

Assume I_1. Integrating by parts and using the equality $q'(\frac{j}{2^n}) = \tilde{q}'(\frac{j}{2^n})$ we obtain

$$\left|\int\limits_0^{\frac{k}{2^n}} (q'(t)-\tilde{q}'(t))e^{p(t)}\,dt\right| \leqslant \left|q(t)-\tilde{q}(t)|_0^{\frac{k}{2^n}}\right| + \left|\int\limits_0^{\frac{k}{2^n}} (q(t)-\tilde{q}(t))(e^{p(t)})'\,dt\right| \leqslant$$

$$\leqslant \omega_{\frac{1}{2^n}}(q)\int\limits_0^{\frac{k}{2^n}} |(e^{p(t)})'|\,dt \leqslant \omega_{\frac{1}{2^n}}(q)\bigvee_0^1 e^{p(\cdot)}.$$

So

$$|I_1| \leqslant \|e^{-p(\cdot)}\|_{C(0,1)}\omega_{\frac{1}{2^n}}(q(\cdot))\bigvee_0^1 e^{p(\cdot)}.$$

Since the function $e^{\tilde{p}(t)}$ is monotonic on any interval $[\frac{j}{2^n}, \frac{j+1}{2^n}]$, it follow that $|(e^{p(t)} - e^{\tilde{p}(t)})| \leqslant \omega_{\frac{1}{2^n}}(e^{p(\cdot)})$. Therefore

$$|I_2| \leqslant \|e^{-p(\cdot)}\| \cdot \omega_{\frac{1}{2^n}}(e^{p(\cdot)})\bigvee_0^1 q(\cdot),$$

and

$$\left|y\left(\frac{k}{2^n}\right) - \tilde{y}\left(\frac{k}{2^n}\right)\right| \leqslant \|e^{-p(\cdot)}\|_{C[0,1]}\left(\omega_{\frac{1}{2^n}}(q(\cdot))\bigvee_0^1 e^{p(\cdot)} + \omega_{\frac{1}{2^n}}(e^{p(\cdot)})\cdot\bigvee_0^1 q(\cdot)\right).$$

2. Now we consider the case $x \in \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]$. Let us write the difference $y(x) - \tilde{y}(x)$ in the form

$$y(x) - \tilde{y}(x) = y_0 e^{p(0)}(e^{-p(x)} - e^{-\tilde{p}(x)}) +$$
$$+(e^{-p(x)} - e^{-\tilde{p}(x)}) \left(\int\limits_0^{\frac{k}{2^n}} q'(t)e^{p(t)}\,dt + \int\limits_{\frac{k}{2^n}}^{x} q'(t)e^{p(t)}\,dt \right) +$$
$$+e^{-\tilde{p}(x)} \left(\int\limits_0^{\frac{k}{2^n}} q'(t)e^{p(t)}\,dt - \int\limits_0^{\frac{k}{2^n}} \tilde{q}'(t)e^{\tilde{p}(t)}\,dt \right) +$$
$$+e^{-\tilde{p}(x)} \left(\int\limits_x^{\frac{k}{2^n}} q'(t)e^{p(t)}\,dt - \int\limits_{\frac{k}{2^n}}^{x} \tilde{q}'(t)e^{\tilde{p}(t)}\,dt \right) = A_1 + A_2 + (A_3 + A_4)e^{-\tilde{p}(x)}. \tag{9}$$

We will estimate A_l $(l = 1, 2, 3, 4)$.

2.1. Since the function $e^{-\tilde{p}(x)}$ is monotonic on any interval $[\frac{j}{2^n}, \frac{j+1}{2^n}]$, it follow that

$$|e^{-p(x)} - e^{-\tilde{p}(x)}| \leqslant \omega_{\frac{1}{2^n}}(e^{-p}). \tag{10}$$

2.2. Using (10) again, we get

$$|A_2| \leqslant \omega_{\frac{1}{2^n}}(e^{-p}) \left| \int\limits_0^{x} q'(t)e^{p(t)}\,dt \right| \leqslant \omega_{\frac{1}{2^n}}(e^{-p}) \|e^p\|_{C[0,1]} \bigvee_0^1 q. \tag{11}$$

2.3. An estimate for A_3 was obtained earlier

$$|A_3| \leqslant \left((\omega_{\frac{1}{2^n}}(q)) \bigvee_0^1 e^p + \omega_{\frac{1}{2^n}}(e^p) \bigvee_0^1 q \right). \tag{12}$$

2.4. Let us write A_4 in the form

$$A_4 = \int\limits_{\frac{k}{2^n}}^{x} q'(t)(e^{p(t)} - e^{\tilde{p}(t)})\,dt + \int\limits_{\frac{k}{2^n}}^{x} e^{\tilde{p}(t)}(q'(t)\tilde{q}')\,dt. \tag{13}$$

Since the function $e^{\tilde{p}(t)}$ is monotonic on the interval $[\frac{k}{2^n}, \frac{k+1}{2^n}]$, both integrals exist. For the first integral, we have the obvious inequality

$$\int\limits_{\frac{k}{2^n}}^{x} q'(t)(e^{p(t)} - e^{\tilde{p}(t)})\,dt \leqslant \omega_{\frac{1}{2^n}}(e^p) \cdot \int\limits_{\frac{k+1}{2^n}}^{\frac{k}{2^n}} |q'(t)|\,dt \leqslant \omega_{\frac{1}{2^n}}(e^p) \bigvee_0^1 q.$$

Integrating by parts the second integral in (13) we have

$$\left| \int\limits_{\frac{k}{2^n}}^{x} e^{\tilde{p}(t)}(q'(t) - \tilde{q}'(t))\,dt \right| \leqslant |q(x) - \tilde{q}(x)| + \omega_{\frac{1}{2^n}}(q) \int\limits_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} |(e^{\tilde{p}(t)})|\,dt \leqslant$$

$$\leqslant \omega_{\frac{1}{2^n}}(q) + \omega_{\frac{1}{2^n}}(q) \left| \int\limits_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} e^{\tilde{p}(t)} \cdot \tilde{p}'(t)\, dt \right| \leqslant$$

$$\leqslant \omega_{\frac{1}{2^n}}(q) + \omega_{\frac{1}{2^n}}(q) \|e^p\|_{C[0,1]} \left| p\left(\frac{k+1}{2^n}\right) - p\left(\frac{k}{2^n}\right) \right| \leqslant \omega_{\frac{1}{2^n}}(q)(1 + \|e^p\|_{C[0,1]} \omega_{\frac{1}{2^n}}(p)).$$

Finally, we obtain

$$|A_4| \leqslant \omega_{\frac{1}{2^n}}(e^p) \bigvee_0^1 q + \omega_{\frac{1}{2^n}}(q)(1 + \|e^p\|_{C[0,1]} \omega_{\frac{1}{2^n}}(p)). \tag{14}$$

Substituting inequalities (10), (11), (12), and (14) in (9) we get

$$|y(x) - \tilde{y}(x)| \leqslant |y_0| e^{p(0)} \omega_{\frac{1}{2^n}}(e^{-p}) + \omega_{\frac{1}{2^n}}(e^{-p}) \|e^p\|_{C[0,1]} \bigvee_0^1 q +$$

$$+ \|e^p\|_{C[0,1]} \left(\omega_{\frac{1}{2^n}}(q) \bigvee_0^1 e^p + \omega_{\frac{1}{2^n}}(e^p) \bigvee_0^1 q \right) +$$

$$+ \|e^p\|_{C[0,1]} \left(\omega_{\frac{1}{2^n}}(e^p) \bigvee_0^1 q + \omega_{\frac{1}{2^n}}(q)(1 + \|e^p\|_{C[0,1]} \omega_{\frac{1}{2^n}}(p)) \right) =$$

$$= C_{-1} \omega_{\frac{1}{2^n}}(e^{-p}) + C_1 \omega_{\frac{1}{2^n}}(e^p) + \omega_{\frac{1}{2^n}}(q) C_0 + C_2 \omega_{\frac{1}{2^n}}(q) \omega_{\frac{1}{2^n}}(p).$$

□

4. Some examples

Example 3. Let us consider the Cauchy problem

$$\begin{cases} y' + \frac{1}{2\sqrt{x}} y = 1 + \frac{1}{\sqrt{x}}, & x \in [0,1], \\ y(0) = 0. \end{cases} \tag{15}$$

Here $p(x) = \sqrt{x}$, $q(x) = x + 2\sqrt{x}$. The solution $y(x) = 2\sqrt{x}$ of this problem is a continuous function on $[0,1]$, but the derivative $y'(0)$ does not exist. We denote by $\tilde{y}(x)$ the approximative solution for some $N = 2^n > 1$. In the Table 1 we give the approximative solution of Cauchy problem (15).

Table 1

The approximative solution of Cauchy problem (15)

x	$y(x)$	$\tilde{y}(x)$ for N			
		$N = 16$	$N = 32$	$N = 64$	$N = 128$
0.000	0.00000	0.00000	0.00000	0.00000	0.00000
0.125	0.70710	0.70485	0.70629	0.70681	0.70700
0.250	1.00000	0.99793	0.99927	0.99974	0.99991
0.500	1.41421	1.41244	1.41359	1.41399	1.41413
0.750	1.73205	1.73050	1.73151	1.73186	1.73198
1.000	2.00000	1.99862	1.99952	1.99983	1.99994

In this table $y(x)$ is the sharp solution, $\tilde{y}(x)$ is the approximative solution.

Example 4. Let us consider the Cauchy problem

$$\begin{cases} y' + p'(x)y = q'(x), & x \in [0,1], \\ y(0) = 0, \end{cases} \tag{16}$$

where

$$p(x) = \begin{cases} \sqrt{x} & \text{if } x \in [0,1/3], \\ (2/3 - x)\sqrt{3} & \text{if } x \in [1/3,2/3], \\ \sqrt{x - 2/3} & \text{if } x \in [2/3,1], \end{cases}$$

$$p'(x) = \begin{cases} \frac{1}{2\sqrt{x}} & \text{if } x \in [0,1/3], \\ -\sqrt{3} & \text{if } x \in [1/3,2/3], \\ \frac{1}{2\sqrt{x-2/3}} & \text{if } x \in [2/3,1], \end{cases}$$

$$q(x) = \begin{cases} \frac{x}{2} + \sqrt{x} - 2/3 & \text{if } x \in [0,1/3], \\ -x(2+\sqrt{3}) + \frac{3}{2}x^2 + \frac{2}{\sqrt{3}} & \text{if } x \in [1/3,2/3], \\ \frac{x}{2} + \sqrt{x - 2/3} - 1 & \text{if } x \in [2/3,1], \end{cases}$$

$$q'(x) = \begin{cases} \frac{1}{2}(1 + \frac{1}{\sqrt{x}}) & \text{if } x \in [0,1/3], \\ -2 - \sqrt{3} + 3x & \text{if } x \in [1/3,2/3], \\ \frac{1}{2}(1 + \frac{1}{\sqrt{x-2/3}}) & \text{if } x \in [2/3,1]. \end{cases}$$

The solution

$$y(x) = \begin{cases} \sqrt{x}, & x \in [0,1/3], \\ \sqrt{3}(2/3 - x), & x \in [1/3,2/3], \\ \sqrt{x - 2/3}, & x \in [2/3,1], \end{cases}$$

of this problem is a continuous function on $[0,1]$, but the derivatives $y'(1/3)$, $y'(2/3)$, $y'(0)$ do not exist.

In the Figure we demonstrate graphs of the approximate and exact solutions. Both graphs are drawn on 512 points. First, we drew a graph of the approximate solution (blue color), then a graph of the exact solution (red color). We see that these graphs coincided.

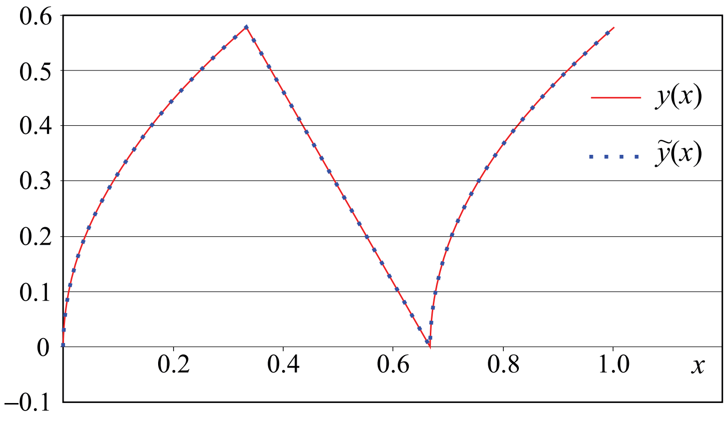

Fig. The graphs of $\tilde{y}(x)$ (blue) and $y(x)$ (red) for $2^n = N = 512$ (color online)

We denote by $\tilde{y}_n(x)$ the approximative solution for the point system $(j2^{-n})_{j=0}^{2^n}$ and $\delta_n = \max_j |\tilde{y}_n(j2^{-n}) - y(j2^{-n})|$. In the Table 2 we give the error of the approximative solution of Cauchy problem (16) for $n = \overline{4, 10}$.

Table 2

The error of the approximative solution for 2^n

n	4	5	6	7	8	9	10
2^n	16	32	64	128	256	512	1024
δ_n	$1.1 \cdot 10^{-3}$	$5.3 \cdot 10^{-4}$	$1.8 \cdot 10^{-4}$	$8.6 \cdot 10^{-5}$	$2.8 \cdot 10^{-5}$	$1.2 \cdot 10^{-5}$	$3.9 \cdot 10^{-6}$

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