Известия Саратовского университета. Новая серия. Серия: Математика. Механика. Информатика. 2021. Т. 21, вып. 3. С. 317–325
Izvestiya of Saratov University. Mathematics. Mechanics. Informatics, 2021, vol. 21, iss. 3, pp. 317–325
https://mmi.sgu.ru https://doi.org/10.18500/1816-9791-2021-21-3-317-325
Article

Solutions of the Loewner equation with combined driving functions

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Abstract. The paper is devoted to the multiple chordal Loewner differential equation with different driving functions on two time intervals. We obtain exact implicit or explicit solutions to the Loewner equations with piecewise constant driving functions and with combined constant and square root driving functions. In both cases, there is an analytical and geometrical description of generated traces. Earlier, Kager, Nienhuis and Kadanoff integrated the chordal Loewner differential equation either with a constant driving function or with a square root driving function. In the first case, the equation generates a rectilinear slit in the upper half-plane which is orthogonal to the real axis $\mathbb{R}$. In the second case, a rectilinear slit forms an angle to $\mathbb{R}$. In our paper, the multiple chordal Loewner differential equation generates more complicated hulls consisting of three rectilinear and curvilinear fragments which can be either intersecting or disjoint. Analytical results of the paper are accompanied by geometrical illustrations.
Keywords: Loewner equation, driving function, trace, integrability case
Acknowledgements: This work was supported by the Program of development of Regional Scientific and Educational Mathematical Center "Mathematics of Future Technologies" (project No. 075-02-2021-1399).
For citation: Prokhorov D. V., Zakharov A. M., Zherdev A. V. Solutions of the Loewner equation with combined driving functions. *Izvestiya of Saratov University. Mathematics. Mechanics. Informatics*, 2021, vol. 21, iss. 3, pp. 317–325 (in English). https://doi.org/10.18500/1816-9791-2021-21-3-317-325

Научная статья
УДК 517.54

О решениях уравнения Лёвнера с составными управляющими функциями

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Аннотация. В статье рассматривается хордовое дифференциальное уравнение Лёвнера с управлением, заданным разными функциями на частях отрезка интегрирования. Получены точные решения в явном или неявном виде для кусочно-постоянной управляющей функции, а также управления, заданного как комбинация постоянной функции и квадратного корня. Для обоих случаев дано аналитическое и геометрическое описания генерируемых разрезов. Ранее Кагер, Ниенуис и Каданов проинтегрировали хордовое дифференциальное уравнение Лёвнера с постоянной управляющей функцией и с управляющей функцией в виде квадратного корня. В первом случае уравнение генерирует в верхней полуплоскости прямолинейный разрез, ортогональный к вещественной оси $\mathbb{R}$. Во втором случае прямолинейный разрез образует некоторый угол с осью $\mathbb{R}$, зависящий от коэффициента при квадратном корне. В настоящей статье обобщенное дифференциальное уравнение Лёвнера генерирует более сложные множества, состоящие из трех прямолинейных или криволинейных фрагментов, которые могут пересекаться или не иметь общих точек. Аналитические результаты статьи сопровождаются геометрическими интерпретациями.

Ключевые слова: уравнение Лёвнера, управляющая функция, разрез, случай интегрируемости

Благодарности: Работа выполнена при поддержке Программы развития регионального научно-образовательного математического центра «Математика технологий будущего» (проект № 075-02-2021-1399).

Для цитирования: *Prokhorov D. V., Zakharov A. M., Zherdev A. V.* Solutions of the Loewner equation with combined driving functions [*Прохоров Д. В., Захаров А. М., Жердев А. В.* О решениях уравнения Лёвнера с составными управляющими функциями] // Известия Саратовского университета. Новая серия. Серия: Математика. Механика. Информатика. 2021. Т. 21, вып. 3. С. 317–325. https://doi.org/10.18500/1816-9791-2021-21-3-317-325

Introduction

The Loewner differential equations [1] play an important role in the geometric function theory of complex analysis. We will discuss a half-plane version of the Loewner equation, see e. g., [2, Chapter 4], generating self-maps of the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. Given a simple curve Γ in $\mathbb{H}$, emanating from a point on $\mathbb{R}$, and for an appropriate continuous parametrization $\Gamma(t)$ of Γ, $0 \leqslant t \leqslant T$, there exists a unique conformal map $g(\cdot, t)$ from $\mathbb{H} \setminus \Gamma[0, t]$ onto $\mathbb{H}$ that obeys the hydrodynamic normalization near infinity

$$g(z,t) = z + \frac{2t}{z} + O\left(\frac{1}{|z|^2}\right), \quad z \to \infty.$$

In this case, there is a continuous driving function $\lambda : [0, T] \to \mathbb{R}$ such that g solves the chordal Loewner differential equation

$$\frac{\partial g(z,t)}{\partial t} = \frac{2}{g(z,t) - \lambda(t)}, \quad g(z,0) = z, \quad 0 \leqslant t \leqslant T, \quad z \in \mathbb{H} \setminus \Gamma[0,T]. \tag{1}$$

We say that g generates Γ.

If Γ is a finite union of simple curves, probably with common points, we need to use the multiple Loewner differential equation

$$\frac{\partial g(z,t)}{\partial t} = \sum_{k=1}^{n} \frac{2\mu_k}{g(z,t)-\lambda_k(t)}, \quad g(z,0)=z, \quad 0\leqslant t\leqslant T, \quad z\in\mathbb{H}\setminus\Gamma[0,T]$$

with (piecewise) continuous driving functions $\lambda_k : [0,T]\to\mathbb{R}$ and positive numbers μ_k, $k=1,\dots,n$, $\sum_{k=1}^{n}\mu_k = 1$.

In this paper, we restrict ourselves to $n=2$ and $\mu_1=\mu_2=\frac{1}{2}$. So we consider the Loewner differential equation

$$\frac{\partial g(z,t)}{\partial t} = \sum_{k=1}^{2} \frac{1}{g(z,t)-\lambda_k(t)}, \quad g(z,0)=z, \quad 0\leqslant t\leqslant T, \quad z\in\mathbb{H}\setminus\Gamma[0,T]. \tag{2}$$

There are only few known examples of driving functions in (1) or (2) that admit explicit integration of this equation and describe corresponding traces Γ. In [3], the authors solve equation (2) with constant driving functions $\lambda_1<0$ and $\lambda_2=-\lambda_1$. For $n=1$, the authors of [3] give a full description of a trace generated in equation (1) if the driving function has the form $\lambda(t)=A\sqrt{t}$, $A>0$.

We aim to develop integration possibilities for equation (2) with combined driving functions λ_1 and λ_2, $\lambda_1=-\lambda_2$, when

$$\lambda_2(t) = \begin{cases} 0, & 0\leqslant t<t_0, \\ A, & t_0\leqslant t\leqslant T, \end{cases} \tag{3}$$

or

$$\lambda_2(t) = \begin{cases} 0, & 0\leqslant t<t_0, \\ A\sqrt{t-t_0}, & t_0\leqslant t\leqslant T \end{cases} \tag{4}$$

for arbitrary $A>0$ and $t_0>0$ and a certain $T>t_0$.

Note that both driving functions λ_1,λ_2 in (4) are continuous on $[0,T]$ while driving functions λ_1,λ_2 in (3) have jumps at t_0.

In Section 1, we integrate Loewner equation (2) with piecewise constant driving functions (3), see Theorem 1, and show that a solution $g(\cdot,t)$ maps $\mathbb{H}\setminus\Gamma$ onto $\mathbb{H}$, where Γ is a union of a segment Γ_0 on the upper imaginary half-axis and a pair of curves Γ_1 and Γ_2 which are symmetric with respect to the imaginary axis and emanate either from points on $\mathbb{R}$ if $A>2\sqrt{t_0}$ or from points on Γ_0 if $A<2\sqrt{t_0}$. If $A=2\sqrt{t_0}$, the boundary symmetric curves Γ_1 and Γ_2 emanate from the origin under angles $\pm\frac{\pi}{4}$ to the real axis $\mathbb{R}$. We give implicit representations of Γ_1 and Γ_2 and asymptotic expansions for Γ_1 and Γ_2 near $t=t_0$.

In Section 2, we integrate Loewner equation (2) with continuous driving functions (4) which are constant on $[0,t_0)$ and square root functions on $[t_0,T]$, see Theorem 2. We show that a solution $g(\cdot,t)$ maps $\mathbb{H}\setminus\Gamma$ onto $\mathbb{H}$, where Γ is a union of the segment $[0,i2\sqrt{t_0}]$ and a pair of curves which are symmetric with respect to the imaginary axis and emanate from the point $i2\sqrt{t_0}$. We give explicit representations of boundary curves and their asymptotic expansions near $t=t_0$.

In Section 4, we discuss an interrelation between exact solutions for the standard Loewner equation on two separate time intervals and its multiple version.

1. Loewner equation with piecewise constant driving functions

Let us solve the multiple Loewner differential equation (2) with combined driving functions (3) that are piecewise constant on $[0,T]$.

Theorem 1. *There exists $T > t_0$ for which the multiple Loewner differential equation (2) with combined driving functions (3) has a solution $w = g(z,t)$ on $[0,T]$. On $[0,t_0]$, $g(z,t) = \sqrt{z^2+4t}$, and on $[t_0,T]$, $w = g(z,t)$ satisfies the implicit equation*

$$w^2 - z^2 - A^2 \log \frac{w^2}{z^2+4t_0} = 4t, \;\; g(z,t_0) = \sqrt{z^2+4t_0}, \quad z \in \mathbb{H} \setminus [0, i2\sqrt{t_0}], \tag{5}$$

where the continuous branches of $\log w$ and $\log z$ are real when w and z are positive. The function $g(z,T)$ maps $\mathbb{H}\setminus\Gamma$ onto $\mathbb{H}$ according to the following three cases:

(i) If $A > 2\sqrt{t_0}$, then $\Gamma = \cup_{k=0}^{2}\Gamma_k$, where $\Gamma_0 = [0, 2i\sqrt{t_0}]$, $\Gamma_2[0,T]$ is a curve which emanates from $\sqrt{A^2-4t_0}$ and is orthogonal to $\mathbb{R}$ at this point, $\Gamma_1[0,T]$ is symmetric to $\Gamma_2[0,T]$ with respect to the imaginary axis (Fig. 1);

(ii) If $A < 2\sqrt{t_0}$, then $\Gamma = \cup_{k=0}^{2}\Gamma_k^*$, where $\Gamma_0^* = \Gamma_0$, $\Gamma_2^*[0,T]$ is a curve which emanates from $i\sqrt{4t_0 - A^2}$ and is orthogonal to the imaginary axis at this point, $\Gamma_1^*[0,T]$ is symmetric to $\Gamma_2^*[0,T]$ with respect to the imaginary axis (Fig. 2);

(iii) If $A = 2\sqrt{t_0}$, then $\Gamma = \cup_{k=0}^{2}\Gamma_k^{**}$, where $\Gamma_0^{**} = \Gamma_0$, $\Gamma_2^{**}[0,T]$ is a curve which emanates from the origin under the angle $\frac{\pi}{4}$ to $\mathbb{R}$, $\Gamma_1^{**}[0,T]$ is symmetric to $\Gamma_2^{**}[0,T]$ with respect to the imaginary axis (Fig. 3).

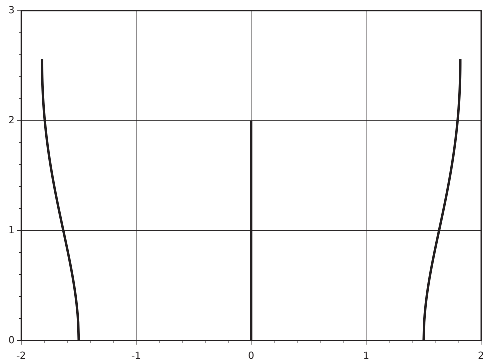

Fig. 1. Γ for $t_0 = 1$, $T = 3$, $A = 2.5$ (i)

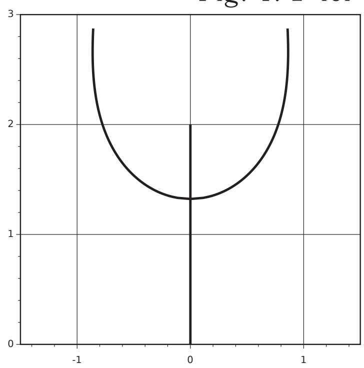

Fig. 2. Γ for $t_0 = 1$, $T = 3$, $A = 1.5$ (ii)

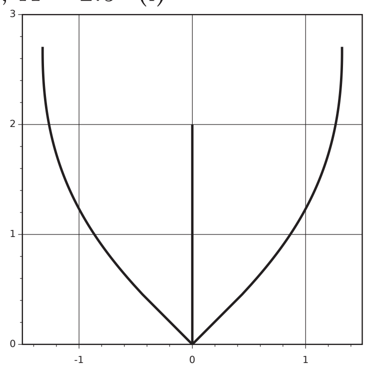

Fig. 3. Γ for $t_0 = 1$, $T = 3$, $A = 2$ (iii)

Proof. It is a well-known result on $[0,t_0]$ that $g(z,t) = \sqrt{z^2+4t}$, $z \in \mathbb{H}$, see, e.g., [2, p. 95], [3]. Next, we have to solve the multiple Loewner equation

$$\frac{dw}{dt} = \frac{1}{w+A} + \frac{1}{w-A} = \frac{2w}{w^2-A^2}, \quad w(z,t_0) = g(z,t_0), \quad t_0 \leqslant t \leqslant T. \tag{6}$$

The function $g(z,t_0)$ maps $\mathbb{H}\setminus[0,i2\sqrt{t_0}]$ onto $\mathbb{H}$. Differential equation (6) with separated variables w and z has a general solution in the form

$$w^2 - 2A^2\log w = 4t + c$$

with an arbitrary constant c. The initial value allows us to determine c as

$$c = z^2 - A^2\log(z^2+4t_0).$$

So we find an implicit solution $w = w(z,t)$ to the Cauchy problem (6) as it is presented in (5).

Differential equation (6) generates two traces Γ_1 and Γ_2 that are symmetric with respect to the imaginary axis and emanate from two points $g(z,t_0) = \pm A$ on $\mathbb{R}$. Let Γ_2 correspond to $g(z,t_0) = A$ and let Γ_2 be given by $z = z(t)$. Then the line of singularities $z(t)$ satisfies the equation

$$w(z(t),t) = A, \quad t \geqslant t_0.$$

Together with (5) this leads to the equality

$$A^2 - z^2(t) - A^2\log\frac{A^2}{z^2(t)+4t_0} = 4t, \quad t \geqslant t_0, \quad g(z(t_0),t_0) = A. \tag{7}$$

The equality $g(z(t_0),t_0) = A$ is equivalent to $z(t_0) = \sqrt{A^2-4t_0}$. A disposition of the initial point of Γ_2 depends on the sign of A^2-4t_0. Let us consider three possible cases.

Case (i): $A > 2\sqrt{t_0}$. Then $z(t_0) > 0$ and Γ_2 emanates from the point on the positive real half-axis.

Case (ii): $0 < A < 2\sqrt{t_0}$. Then $z(t_0)$ is pure imaginary and Γ_2 emanates from the point on $(0, i2\sqrt{t_0})$.

Case (iii): $A = 2\sqrt{t_0}$. Then $z(t_0) = 0$ and Γ_2 emanates from the origin.

Equality (7) is an implicit representation of Γ_2. Find an asymptotic expansion of Γ_2 near the initial point in all the three cases.

Differentiate (7) and obtain

$$(z^2(t))' = \frac{4(z^2(t)+4t_0)}{A^2-4t_0-z^2(t)}, \quad z^2(t_0) = A^2-4t_0, \quad t \geqslant t_0. \tag{8}$$

This allows us to find an asymptotic expansion for $z(t)$ near t_0. In cases (i) and (ii) it is reasonable to set

$$z(t) = \sqrt{A^2-4t_0} + a\sqrt{t-t_0} + o(\sqrt{t-t_0}), \quad t \to t_0^+.$$

Hence

$$(z^2(t))' = \frac{a\sqrt{A^2-4t_0}}{\sqrt{t-t_0}} + o\left(\frac{1}{\sqrt{t-t_0}}\right), \quad t \to t_0^+.$$

Substitute expansions for $z(t)$ and $(z^2(t))'$ in (8) and see that

$$\frac{a\sqrt{A^2-4t_0}}{\sqrt{t-t_0}} = -\frac{2A^2}{a\sqrt{A^2-4t_0}\sqrt{t-t_0}} + o\left(\frac{1}{\sqrt{t-t_0}}\right), \quad t \to t_0^+,$$

which gives that

$$a^2 = -\frac{2A^2}{A^2-4t_0}.$$

In Case (i) $A^2 > 4t_0$:

$$z(t) = \sqrt{A^2 - 4t_0} + i\frac{\sqrt{2}A}{\sqrt{A^2 - 4t_0}}\sqrt{t - t_0} + o(\sqrt{t - t_0}), \quad t \to t_0^+.$$

So $z(t)$ is orthogonal to $\mathbb{R}$ at $z = \sqrt{A^2 - 4t_0}$.

In Case (ii) $A^2 < 4t_0$:

$$z(t) = i\sqrt{4t_0 - A^2} + \frac{\sqrt{2}A}{\sqrt{4t_0 - A^2}}\sqrt{t - t_0} + o(\sqrt{t - t_0}), \quad t \to t_0^+.$$

So $z(t)$ is orthogonal to the imaginary axis at $z = i\sqrt{4t_0 - A^2}$.

Case (iii) $A^2 = 4t_0$ requires another asymptotic behavior of the trace near the origin. Formula (8) transforms to the following

$$(z^4(t))' = -8(z^2(t) + 4t_0), \quad z(t_0) = 0.$$

Let us present another reasonable asymptotic expansion for $z(t)$,

$$z(t) = b\sqrt[4]{t - t_0} + o(\sqrt[4]{t - t_0}), \quad t \to t_0^+.$$

We take into account both last formulas and obtain that $b^4 = -32t_0$. Thus

$$z(t) = e^{i\frac{\pi}{4}} 2\sqrt[4]{2t_0}\sqrt[4]{t - t_0} + o(\sqrt[4]{t - t_0}), \quad t \to t_0^+.$$

So $z(t)$ is tangential to the radial ray at the angle $\frac{\pi}{4}$ to $\mathbb{R}$ from the origin.

Similarly to (7), derive an implicit representation for $z(t)$ in Case (iii). Integrate the differential equation for $(z^4(t))'$ to get the needed equation

$$z^2 + 4\log\frac{A^2}{z^2 + 4t_0} = A^2 - 4t, \quad z(t_0) = 0.$$

The boundary curve Γ_1 can be studied similarly. However, it is symmetric to Γ_1 with respect to the imaginary axis due to the symmetric disposition of points $\pm A$ and the symmetric trace on the time segment $[0, t_0]$.

It remains to be observed what happens to the boundary $[0, i2\sqrt{t_0}]$ when t varies along $[t_0, T]$. The implicit representation (5) implies that the two singular points $w = 0$ and $z = i2\sqrt{t_0}$ appear simultaneously. As far as $w = 0$ is constant on $[t_0, T]$ according to (6), the corresponding $z = i2\sqrt{t_0}$ also does not move for t on $[t_0, T]$. Inner points of the segment $[0, i2\sqrt{t_0}]$ cannot leave the imaginary axis because of symmetric properties of conformal mappings generated by symmetric driving functions (3). This means that the segment $[0, i2\sqrt{t_0}]$ is a part of the boundary set Γ, and there are no additional parts of Γ on the imaginary axis, which completes the proof of Theorem 1. □

2. Combined constant and square root driving functions

Now we will solve the multiple Loewner differential equation (2) with combined driving functions (4) that are continuous on $[0, T]$.

Theorem 2. *For every $T > t_0$, the multiple Loewner differential equation (2) with combined driving functions (4) has a solution $w = g(z, t)$ on $[0, T]$. On $[0, t_0]$, $g(z, t) = \sqrt{z^2 + 4t}$, and on $[t_0, T]$, $w = g(z, t)$ satisfies the implicit equation*

$$(A^2 + 4)(t - t_0) = w^2 - (z^2 + 4t_0)^{\frac{A^2}{4}+1} w^{-\frac{A^2}{2}}, \quad w(z, t_0) = \sqrt{z^2 + 4t_0},$$

where the branches of power functions are such that they are positive when z^2+4t_0 and w are positive. The function $g(z,T)$ maps $\mathbb{H}\setminus\Gamma$ onto $\mathbb{H}$, $\Gamma=\cup_{k=0}^{2}\Gamma_k$, where Γ_0 is the segment $[0,i2\sqrt{t_0}]$, Γ_2 is a square root of a rectilinear segment under the angle $4\pi/(A^2+4)$ to $\mathbb{R}$ from $(-4t_0)$, and Γ_1 is symmetric to Γ_2 with respect to the imaginary axis (Fig. 4).

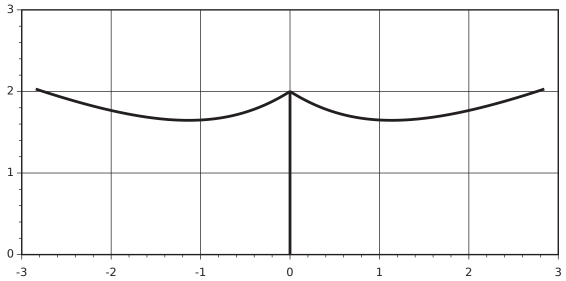

Fig. 4. Γ for $t_0=1,\ T=3,\ A=3.0$

Proof. As in Theorem 1, on $[0,t_0]$, the function $g(z,t)=\sqrt{z^2+4t}$, $z\in\mathbb{H}$, solves the chordal Loewner differential equation (2) with vanishing driving functions. Next, we have to solve the multiple Loewner equation

$$\frac{dw}{dt}=\frac{1}{w+A\sqrt{t-t_0}}+\frac{1}{w-A\sqrt{t-t_0}}=\frac{2w}{w^2-A^2(t-t_0)},\quad t_0\leqslant t\leqslant T \tag{9}$$

with the initial condition $w(z,t_0)=g(z,t_0)$. We keep in mind that $g(z,t_0)=\sqrt{z^2+4t_0}$ maps $\mathbb{H}\setminus[0,2i\sqrt{t_0}]$ onto $\mathbb{H}$.

Note that differential equation (9) is linear with respect to t. Its general solution is given implicitly by

$$t-t_0=\frac{w^2}{A^2+4}+cw^{-\frac{A^2}{2}}$$

with an arbitrary constant c. The initial value allows us to determine c from the equation

$$0=\frac{z^2+4t_0}{A^2+4}+c(z^2+4t_0)^{-\frac{A^2}{4}}$$

so that

$$c=-\frac{(z^2+4t_0)^{\frac{A^2}{4}+1}}{A^2+4}.$$

So we find an implicit solution to the Cauchy problem (9) as

$$t-t_0=\frac{w^2}{A^2+4}-\frac{(z^2+4t_0)^{\frac{A^2}{4}+1}}{A^2+4}w^{-\frac{A^2}{2}}, \tag{10}$$

which proves the first statement of Theorem 2 for a certain $T>t_0$.

Differential equation (9) generates two traces Γ_1 and Γ_2 symmetric with respect to the imaginary axis and emanating from the common point $g(0,t_0)=i2\sqrt{t_0}$ on the imaginary axis. Let Γ_2 be situated in the right half-plane for a certain $T>t_0$ and let Γ_2 be given by $z(t)$. Then the line of singularities $z(t)$ satisfies the equation

$$w(z(t),t)=A\sqrt{t-t_0},\quad t\geqslant t_0.$$

Together with (10) this leads to the equality

$$t - t_0 = \frac{A^2(t-t_0)}{A^2+4} - \frac{(z^2(t)+4t_0)^{\frac{A^2}{4}+1}}{A^2+4}(A\sqrt{t-t_0})^{-\frac{A^2}{2}}, \quad t \geqslant t_0.$$

Transform this expression to the following

$$4(t-t_0)^{\frac{A^2}{4}+1} = -(z^2(t)+4t_0)^{\frac{A^2}{4}+1}A^{-\frac{A^2}{2}}$$

and give the explicit formula for $z(t)$:

$$z(t) = \left[e^{\frac{i4\pi}{A^2+4}}2^{\frac{8}{A^2+4}}A^{\frac{2A^2}{A^2+4}}(t-t_0) - 4t_0\right]^{\frac{1}{2}}, \quad t \geqslant t_0.$$

It is worth noting that $z(t)$ is the square root of a rectilinear segment under the angle $4\pi/(A^2+4)$ to $\mathbb{R}$ from $(-4t_0)$. The slope of the rectilinear segment is changing from π to 0 when A is growing from 0 to infinity. Therefore, Γ_2 emanates from $i2\sqrt{t_0}$ and is tangential to the ray at the angle

$$\frac{\pi 4}{A^2+4} - \frac{\pi}{2} = \frac{\pi(4-A^2)}{2(A^2+4)}$$

to $\mathbb{R}$ at this endpoint. The slope of Γ_2 is changing from $\pi/2$ to $(-\pi/2)$ when A is growing from 0 to infinity.

The last reasoning explains that Γ_2 stays in the right half-plane for all A and $T > t_0$ and it is a simple curve.

The boundary curve Γ_1 can be studied similarly. However, it is symmetric to Γ_1 with respect to the imaginary axis due to the symmetric properties of the driving functions $\pm A\sqrt{t-t_0}$ and the symmetric trace on the time segment $[0, t_0]$.

It is known that the Loewner differential equation generates simple traces up to the moment t when either lines of singularities Γ meet the real axis $\mathbb{R}$ or Γ has self-intersection, see, e.g., [4]. We showed that, under conditions of Theorem 2, the curve Γ_2 stays in the right half-plane for all t and does not reach $\mathbb{R}$. Similarly, the curve Γ_1 stays in the left half-plane and does not reach $\mathbb{R}$. Both Γ_2 and Γ_1 do not meet $\Gamma_0 := [0, i2\sqrt{t_0}]$. Hence the Loewner generating process develops in time for all $T > t_0$. This completes the proof of Theorem 2. □

Conclusions

The proofs of Theorems 1 and 2 are based on the knowledge of integrability cases of the Loewner differential equation for constant and square root driving functions both in the standard and multiple versions. There are some more known driving functions that admit explicit or implicit integration of the Loewner equation. Therefore, it is possible to present new examples of combined driving functions in the Loewner equation with several contact points which join different driving functions and lead to exact solutions.

Point out at such examples. Besides constant and square root driving functions, Kager, Nienhuis and Kadanoff [3] considered linear driving functions and obtained exact solutions of the Loewner equation. We have to add that their adaptation to the multiple equation is not so successful in getting exact solutions.

In [5], the authors found an implicit exact solution of the Loewner equation with the exponential driving function $A(e^t - 1)$. Moreover, this driving function is well-adapted to express an exact solution for the multiple Loewner equation.

There is another approach in the exact solution problem when driving functions are determined for given traces of the Loewner equation. We refer to [6], where the problem was solved for the circular arc in $\mathbb{H}$ tangential to $\mathbb{R}$ at 0. This result was generalized in [7] for powers of this arc and in [8] for tangential curves close to this arc. It was proved in [6] that the tangential circular arc of radius 1 and centered at i is generated by the driving function $\lambda(t) = 3\alpha(t) + 2\sqrt{-\alpha(t)\pi}$, where $\alpha = \alpha(t)$ is an algebraic function satisfying the equation

$$\alpha(3\alpha + 4\sqrt{-\alpha\pi}) = -6t, \quad t \geqslant 0.$$

A similar problem was solved by Wu in [9] for circular arcs in $\mathbb{H}$ which meet $\mathbb{R}$ orthogonally.

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Поступила в редакцию / Received 22.03.2020
Принята к публикации / Accepted 27.04.2021
Опубликована / Published 31.08.2021