Известия Саратовского университета. Новая серия. Серия: Математика. Механика. Информатика. 2021. Т. 21, вып. 3. С. 343–351
Izvestiya of Saratov University. Mathematics. Mechanics. Informatics, 2021, vol. 21, iss. 3, pp. 343–351
https://mmi.sgu.ru https://doi.org/10.18500/1816-9791-2021-21-3-343-351

Article

Differential operators on graphs with a cycle

V. A. Yurko

Saratov State University, 83 Astrakhanskaya St., Saratov 410012, Russia
Vjacheslav A. Yurko, yurkova@sgu.ru, https://orcid.org/0000-0002-4853-0102

Abstract. An inverse problem of spectral analysis is studied for Sturm – Liouville differential operators on a graph with a cycle. We pay the main attention to the most important nonlinear inverse problem of recovering coefficients of differential equations provided that the structure of the graph is known a priori. We use the standard matching conditions in the interior vertices and Robin boundary conditions in the boundary vertices. For this class of operators properties of spectral characteristics are established, a constructive procedure is obtained for the solution of the inverse problem of recovering coefficients of differential operators from spectra, and the uniqueness of the solution is proved. For solving this inverse problem we use the method of spectral mappings, which allows one to construct the potential on each fixed edge. For transition to the next edge we use a special representation of the characteristic functions.
Keywords: Sturm – Liouville operators, geometrical graphs, inverse spectral problems
Acknowledgements: This work was supported in part by the Russian Foundation for Basic Research (project No. 19-01-00102).
For citation: Yurko V. A. Differential operators on graphs with a cycle. *Izvestiya of Saratov University. Mathematics. Mechanics. Informatics*, 2021, vol. 21, iss. 3, pp. 343–351 (in English). https://doi.org/10.18500/1816-9791-2021-21-3-343-351

Научная статья
УДК 539.374

Дифференциальные операторы на графе с циклом

В. А. Юрко

Саратовский национальный исследовательский государственный университет имени Н. Г. Чернышевского, Россия, 410012, г. Саратов, ул. Астраханская, д. 83

Юрко Вячеслав Анатольевич, доктор физико-математических наук, заведующий кафедрой математической физики и вычислительной математики, yurkova@sgu.ru, https://orcid.org/0000-0002-4853-0102

Аннотация. Исследуется обратная задача спектрального анализа для дифференциальных операторов Штурма – Лиувилля на графе с циклом. Основное внимание уделяется наиболее важной нелинейной обратной задаче восстановления коэффициентов дифференциальных уравнений при условии, что структура графа известна априори. Используются стандартные

условия склейки во внутренних вершинах и краевые условия Робина в граничных вершинах. Для данного класса операторов установлены свойства спектральных характеристик, получена конструктивная процедура решения обратной задачи восстановления коэффициентов дифференциальных операторов по спектрам и доказана единственность решения. Для решения этой обратной задачи используется метод спектральных отображений, который позволяет строить потенциал на каждом фиксированном ребре. Для перехода к следующему ребру используется специальное представление характеристических функций.
Ключевые слова: операторы Штурма – Лиувилля, геометрические графы, обратные спектральные задачи

Благодарности: Исследование выполнено при частичной финансовой поддержке РФФИ (проект № 19-01-00102).

Для цитирования: *Yurko V. A.* Differential operators on graphs with a cycle [*Юрко В. А.* Дифференциальные операторы на графе с циклом] // Известия Саратовского университета. Новая серия. Серия: Математика. Механика. Информатика. 2021. Т. 21, вып. 3. С. 343–351. https://doi.org/10.18500/1816-9791-2021-21-3-343-351

Introduction

We study the inverse spectral problem for Sturm – Liouville differential operators on a graph with a cycle and with standard matching conditions in the internal vertex. Inverse spectral problems consist in recovering operators from their spectral characteristics. The main results on inverse problems for differential operators on an *interval* are presented in [1, 2]. Inverse spectral problems for differential operators on graphs were studied in many works (see the review paper [3] and the references therein). In this paper, we obtain the solution of inverse spectral problems of recovering potentials of Sturm – Liouville operators on a graph with a cycle from the given spectral characteristics and prove the uniqueness of the solution.

Consider a compact graph T in $\mathbf{R}^\mathbf{m}$ with vertices $V = \{v_0, \dots, v_r\}$ and edges $\mathscr{E} = \{e_0, \dots, e_r\}$, where $v_1, \dots, v_r$ are the boundary vertices, v_0 is the internal vertex, $e_j = [v_j, v_0]$, $j = \overline{1, r}$, $\bigcap\limits_{j=0}^{r} e_j = \{v_0\}$, and e_0 is a cycle. Thus, the graph T has one cycle e_0 and one internal vertex v_0. Let T_j, $j = \overline{0, r}$, be the length of the edge e_j. Each edge $e_j \in \mathscr{E}$ is parameterized by the parameter $x_j \in [0, T_j]$. It is convenient for us to choose the following orientation: for $j = \overline{1, r}$, the vertex v_j corresponds to $x_j = 0$, and the vertex v_0 corresponds to $x_j = T_j$; for $j = 0$, both ends $x_0 = +0$ and $x_0 = T_0 - 0$ correspond to v_0. An integrable function Y on T may be represented as $Y = \{y_j\}_{j=\overline{0,r}}$, where the function $y_j(x_j)$, $x_j \in [0, T_j]$, is defined on the edge e_j. Let $q = \{q_j\}_{j=\overline{0,r}}$ be an integrable real-valued function on T; q is called the potential. Consider the following differential equation on T:

$$- y_j''(x_j) + q_j(x_j) y_j(x_j) = \lambda y_j(x_j), \quad j = \overline{0, r}, \tag{1}$$

where λ is the spectral parameter, the functions y_j, y_j', $j = \overline{0, r}$, are absolutely continuous on $[0, T_j]$ and satisfy the following matching conditions in the internal vertex v_0:

$$y_0(0) = y_j(T_j), \quad j = \overline{0, r}, \qquad y_0'(0) = \sum_{j=0}^{r} y_j'(T_j), \tag{2}$$

Matching conditions (2) are called the standard conditions. In electrical circuits, (2) expresses Kirchhoff's law; in elastic string network, it expresses the balance of tension, and so on.

Let us consider the boundary value problem B_0 on T for equation (1) with the matching conditions (2) and with the following boundary conditions at the boundary vertices $v_1, \dots, v_r$:

$$U_j(Y) = 0, \quad j = \overline{1,r},$$

where $U_j(Y) := y_j'(0) - b_j y_j(0)$, and $b = \{b_j\}_{j=\overline{1,r}}$ is a real vector. Moreover, we also consider the boundary value problems B_k, $k = \overline{1,r}$, on T for equation (1) with the matching conditions (2) and with the boundary conditions

$$y_k(0) = 0, \quad U_j(Y) = 0, \quad j = \overline{1,r} \setminus k.$$

Denote by $\Lambda_k := \{\lambda_{kn}\}_{n\geqslant 0}$ the eigenvalues (with multiplicities) of the problem B_k, $k = \overline{0,r}$. In contrast to the case of trees, here the specification of the spectra Λ_k, $k = \overline{0,r}$ does not uniquely determine the potential, and we need additional information. Let $S_j(x_j,\lambda)$, $C_j(x_j,\lambda)$, $j = \overline{0,r}$ be the solutions of equation (1) on the edge e_j with the initial conditions $S_j(0,\lambda) = C_j'(0,\lambda) = 0$, $S_j'(0,\lambda) = C_j(0,\lambda) = 1$. For each fixed $x_j \in [0,T_j]$, the functions $S_j^{(\nu)}(x_j,\lambda)$, $C_j^{(\nu)}(x_j,\lambda)$, $j = \overline{0,r}$, $\nu = 0,1$ are entire in λ of order $1/2$. Moreover,

$$\langle C_j(x_j,\lambda), S_j(x_j,\lambda)\rangle \equiv 1,$$

where $\langle y,z\rangle := yz' - y'z$ is the Wronskian of y and z. Let $\varphi_j(x_j,\lambda) := C_j(x_j,\lambda) + b_j S_j(x_j,\lambda)$, $j = \overline{1,r}$. Denote $h(\lambda) := S_0(T_0,\lambda)$, $H(\lambda) := C_0(T_0,\lambda) - S_0'(T_0,\lambda)$. Let $\{\nu_n\}_{n\geqslant 1}$ be zeros of the entire function $h(\lambda)$, and let $\omega_n := \operatorname{sign} H(\nu_n)$, $\Omega = \{\omega_n\}_{n\geqslant 1}$. The inverse problem is formulated as follows.

Inverse problem 1. Given Λ_k, $k = \overline{0,r}$ and Ω, construct the potential q on T and the vector b.

Let us formulate the uniqueness theorem for the solution of Inverse Problem 1. For this purpose together with q, b we consider a pair $\tilde q, \tilde b$. Everywhere below if a symbol α denotes an object related to q, b, then $\tilde\alpha$ will denote the analogous object related to $\tilde q, \tilde b$.

Theorem 1. *If $\Lambda_k = \tilde\Lambda_k$, $k = \overline{0,r}$, and $\Omega = \tilde\Omega$, then $q = \tilde q$ and $b = \tilde b$. Thus, the specification of Λ_k, $k = \overline{0,r}$ and Ω uniquely determines the potential q on T and the vector b.*

This theorem will be proved in section 3. Moreover, we will give a constructive procedure for the solution of Inverse Problem 1. In section 2 we introduce the main notions and prove some auxiliary propositions.

1. Auxiliary propositions

Fix $k = \overline{1,r}$. Let $\Phi_k = \{\Phi_{kj}\}_{j=\overline{0,r}}$ be the solution of equation (1) satisfying (2) and boundary conditions

$$U_j(\Phi_k) = \delta_{jk}, \quad j = \overline{1,r}, \tag{3}$$

where δ_{jk} is the Kronecker symbol. Put $M_k(\lambda) := \Phi_{kk}(0,\lambda)$, $k = \overline{1,r}$. The function $M_k(\lambda)$ is called *the Weyl-type function* with respect to the boundary vertex v_k. Clearly,

$$\Phi_{kk}(x_k,\lambda) = S_k(x_k,\lambda) + M_k(\lambda)\varphi_k(x_k,\lambda), \quad x_k \in [0,T_k], \quad k = \overline{1,r}, \tag{4}$$

and consequently, $\langle \varphi_k(x_k,\lambda),\, \Phi_{kk}(x_k,\lambda)\rangle \equiv 1$. Using the fundamental system of solutions $\varphi_j(x_j,\lambda)$, $S_j(x_j,\lambda)$, we get

$$\Phi_{kj}(x_j,\lambda) = M^1_{kj}(\lambda)S_j(x_j,\lambda) + M^0_{kj}(\lambda)\varphi_j(x_j,\lambda), \quad x_j \in [0,T_j], \quad j=\overline{0,r}, \quad k=\overline{1,r}. \tag{5}$$

In particular, $M^1_{kk}(\lambda)=1$, $M^0_{kk}(\lambda)=M_k(\lambda)$. Substituting (5) into (2) and (3) we obtain a linear algebraic system s_k with respect to $M^\nu_{kj}(\lambda)$, $\nu=0,1$, $j=\overline{0,r}$. The determinant $\Delta_0(\lambda)$ of s_k does not depend on k and has the form

$$\Delta_0(\lambda) = (d(\lambda)-2)\prod_{j=1}^{r}\varphi_j(T_j,\lambda) + D(\lambda)h(\lambda), \tag{6}$$

where

$$d(\lambda) = C_0(T_0,\lambda) + S_0'(T_0,\lambda), \tag{7}$$

$$D(\lambda) = \sum_{i=1}^{r}\varphi_i'(T_i,\lambda)\prod_{j=1,\, j\neq i}^{r}\varphi_j(T_j,\lambda). \tag{8}$$

The function $\Delta_0(\lambda)$ is entire in λ of order $1/2$, and its zeros coincide with the eigenvalues of the boundary value problem B_0. Solving the algebraic system s_k by Cramer's rule, we get $M^s_{kj}(\lambda) = \Delta^s_{kj}(\lambda)/\Delta_0(\lambda)$, $s=0,1$, $j=\overline{0,r}$, where the determinant $\Delta^s_{kj}(\lambda)$ is obtained from $\Delta_0(\lambda)$ by the replacement of the column which corresponds to $M^s_{kj}(\lambda)$ with the column of free terms. In particular,

$$M_k(\lambda) = -\frac{\Delta_k(\lambda)}{\Delta_0(\lambda)}, \quad k=\overline{1,r}, \tag{9}$$

where

$$\Delta_k(\lambda) = (d(\lambda)-2)S_k(T_k,\lambda)\prod_{j=1,\, j\neq k}^{r}\varphi_j(T_j,\lambda) + D_k(\lambda)h(\lambda), \tag{10}$$

$$D_k(\lambda) = S_k'(T_k,\lambda)\prod_{j=1,\, j\neq k}^{r}\varphi_j(T_j,\lambda) + S_k(T_k,\lambda)\sum_{i=1,\, i\neq k}^{r}\varphi_i'(T_i,\lambda)\prod_{j=1,\, j\neq i,k}^{r}\varphi_j(T_j,\lambda). \tag{11}$$

We note that $\Delta_k(\lambda)$ is obtained from $\Delta_0(\lambda)$ by the replacement of $\varphi_k^{(\nu)}(T_k,\lambda)$, $\nu=0,1$, with $S_k^{(\nu)}(T_k,\lambda)$, $\nu=0,1$. The function $\Delta_k(\lambda)$ is entire in λ of order $1/2$, and its zeros coincide with the eigenvalues of the boundary value problem B_k. The functions $\Delta_k(\lambda)$ are called the characteristic functions for the boundary value problems B_k.

Let us now study the asymptotic behavior of solutions of equation (1). Let $\lambda=\rho^2$, $\operatorname{Im}\rho \geqslant 0$. Denote $\Lambda := \{\rho : \ \operatorname{Im}\rho \geqslant 0\}$, $\Lambda^\delta := \{\rho : \ \arg\rho \in [\delta, \pi-\delta]\}$. For each fixed $j=\overline{0,r}$ on the edge e_j, there exists a fundamental system of solutions of equation (1) $\{e_{j1}(x_j,\rho), e_{j2}(x_j,\rho)\}$, $x_j\in[0,T_j]$, $\rho\in\Lambda$, $|\rho|\geqslant\rho^*$ with the properties:

1) the functions $e^{(\nu)}_{js}(x_j,\rho)$, $\nu=0,1$, are continuous for $x_j\in[0,T_j]$, $\rho\in\Lambda$, $|\rho|\geqslant\rho^*$;

2) for each fixed $x_j\in[0,T_j]$, the functions $e^{(\nu)}_{js}(x_j,\rho)$, $\nu=0,1$, are analytic for $\operatorname{Im}\rho>0$, $|\rho|>\rho^*$;

3) uniformly in $x_j\in[0,T_j]$, the following asymptotical formulae hold

$$e^{(\nu)}_{j1}(x_j,\rho) = (i\rho)^\nu \exp(i\rho x_j)[1], \quad e^{(\nu)}_{j2}(x_j,\rho) = (-i\rho)^\nu \exp(-i\rho x_j)[1], \quad \rho\in\Lambda, \ |\rho|\to\infty, \tag{12}$$

where $[1] = 1 + O(\rho^{-1})$.

Fix $k = \overline{1,r}$. One has

$$\Phi_{kj}(x_j,\lambda) = A^1_{kj}(\rho)e_{j1}(x_j,\rho) + A^0_{kj}(\rho)e_{j2}(x_j,\rho), \quad x_j \in [0,T_j]. \tag{13}$$

Substituting (13) into (2) and (3) we obtain a linear algebraic system s_k^0 with respect to $A^\nu_{kj}(\rho)$, $\nu = 0,1$, $j=\overline{0,r}$. The determinant $\delta_0(\rho)$ of s_k^0 does not depend on k and has the form

$$\delta_0(\rho) = 2^{r+1}\Delta_0(\lambda), \quad \rho\in\Lambda, \quad |\rho|>\rho^*.$$

Moreover,

$$\delta_0(\rho) = (r+2)\exp\Big(-i\rho\sum_{j=0}^r T_j\Big)[1], \quad \rho\in\Lambda^\delta, \quad |\rho|\to\infty. \tag{14}$$

Solving the algebraic system s_k^0 by Cramer's rule and using (12) and (14), we get

$$A^1_{kk}(\rho)=[1], \quad A^0_{kk}(\rho) = \frac{a_k}{i\rho}\exp(2i\rho T_k)[1], \quad \rho\in\Lambda^\delta, \quad |\rho|\in\infty,$$

where a_k is a constant. Together with (12) and (13) this yields for each fixed $x_k\in[0,T_k)$:

$$\Phi^{(\nu)}_{kk}(x_k,\lambda) = (i\rho)^{\nu-1}\exp(i\rho x_k)[1], \quad \rho\in\Lambda^\delta, \quad |\rho|\to\infty. \tag{15}$$

In particular, $M_k(\lambda) = (i\rho)^{-1}[1]$, $\rho\in\Lambda^\delta$, $|\rho|\to\infty$. Moreover, uniformly in $x_j\in[0,T_j]$,

$$S_j^{(\nu)}(x_j,\lambda) = \frac{1}{2i\rho}\Big((i\rho)^\nu\exp(i\rho x_j)[1] - (-i\rho)^\nu\exp(-i\rho x_j)[1]\Big), \quad \rho\in\Lambda, \quad |\rho|\to\infty, \tag{16}$$

$$C_j^{(\nu)}(x_j,\lambda) = \frac{1}{2}\Big((i\rho)^\nu\exp(i\rho x_j)[1] + (-i\rho)^\nu\exp(-i\rho x_j)[1]\Big), \quad \rho\in\Lambda, \quad |\rho|\to\infty. \tag{17}$$

Let us now we study properties of the spectra and the characteristic functions of B_k. Let $\lambda^0_{kn} = (\rho^0_{nk})^2$, $k=\overline{0,r+1}$, be the eigenvalues of the boundary value problems B_k with the zero potential $q=0$ and $b=0$. Let $\Delta_k^0(\lambda)$ be the corresponding characteristic functions. According to (6)–(8), (14) and (10)–(11) we have

$$\Delta_0^0(\lambda) = 2(\cos\rho T_0 - 1)\prod_{j=1}^r\cos\rho T_j - \sin\rho T_0\sum_{i=1}^r\sin\rho T_i\prod_{j=1,\,j\ne i}^r\cos\rho T_j, \tag{18}$$

$$\Delta_k^0(\lambda) = 2(\cos\rho T_0 - 1)\frac{\sin\rho T_k}{\rho}\prod_{j=1,\,j\ne k}^r\cos\rho T_j +$$

$$+\frac{\sin\rho T_0}{\rho}\Big(\cos\rho T_k\prod_{j=1,\,j\ne k}^r\cos\rho T_j - \sin\rho T_k\sum_{i=1,\,i\ne k}^r\sin\rho T_i\prod_{j=1,\,j\ne i,k}^r\cos\rho T_j\Big). \tag{19}$$

Denote $\tau := \operatorname{Im}\rho$. It follows from (6)–(8), (10)–(11), (16) and (17) that for $|\rho|\to\infty$,

$$\Delta_0(\lambda) = \Delta_0^0(\lambda) + O\Big(|\rho|^{-1}\exp\Big(|\tau|\sum_{j=0}^r T_j\Big)\Big),$$

$$\Delta_k(\lambda) = \Delta_k^0(\lambda) + O\Big(|\rho|^{-2}\exp\Big(|\tau|\sum_{j=0}^r T_j\Big)\Big), \quad k=\overline{1,r}. \tag{20}$$

Using (18)–(20), by the well-known method (see, for example, [4]), one can obtain the following properties of the characteristic functions $\Delta_k(\lambda)$ and the eigenvalues Λ_k of the boundary value problems B_k, $k=\overline{0,r}$.

1. For $\rho\in\Lambda$, $|\rho|\to\infty$,

$$\Delta_0(\lambda)=O\Big(\exp\Big(|\tau|\sum_{j=0}^{r}T_j\Big)\Big),$$

$$\Delta_k(\lambda)=O\Big(|\rho|^{-1}\exp\Big(|\tau|\sum_{j=0}^{r}T_j\Big)\Big),\quad k=\overline{1,r}.$$

2. There exist $h>0$ and $C_h>0$ such that

$$|\Delta_0(\lambda)|\geqslant C_h\exp\Big(|\tau|\sum_{j=0}^{r}T_j\Big),$$

$$|\Delta_k(\lambda)|\geqslant C_h|\rho|^{-1}\exp\Big(|\tau|\sum_{j=0}^{r}T_j\Big),\quad k=\overline{1,r}$$

for $|\tau|\geqslant h$. Hence, the eigenvalues $\lambda_{kn}=\rho_{nk}^2$ lie in the domain $|\operatorname{Im}\rho|<h$.

3. The number $N_{\xi k}$ of zeros of $\Delta_k(\lambda)$ in the rectangle $\Pi_\xi=\{\rho:\ |\operatorname{Im}\rho|\leqslant h,$ $\operatorname{Re}\rho\in[\xi,\xi+1]\}$ is bounded with respect to ξ.

4. Denote $G_\delta=\{\rho:\ |\rho-\rho_{0n}|\geqslant\delta\ \ \forall\ n\geqslant 0\}$, $\delta>0$. Then

$$|\Delta_0(\lambda)|\geqslant C_\delta\exp\Big(|\tau|\sum_{j=0}^{r}T_j\Big),\quad \rho\in G_\delta.$$

5. There exist numbers $r_N\to\infty$ such that for sufficiently small $\delta>0$, the circles $|\rho|=r_N$ lie in G_δ for all N.

6. For $n\to\infty$,

$$\rho_{nk}=\rho_{nk}^0+O\Big(\frac{1}{\rho_{nk}^0}\Big).$$

Now the reconstruction of the characteristic functions from their zeros is studied. Denote

$$\lambda_{kn}^{01}=\begin{cases}\lambda_{kn}^0 & \text{if}\quad \lambda_{kn}^0\neq 0,\\ 1 & \text{if}\quad \lambda_{kn}^0=0.\end{cases}\tag{21}$$

By Hadamard's factorization theorem,

$$\Delta_k^0(\lambda)=A_k^0\prod_{n=0}^{\infty}\frac{\lambda_{kn}^0-\lambda}{\lambda_{kn}^{01}},\tag{22}$$

where

$$A_k^0=(-1)^{s_k}\Big(\frac{\partial^{s_k}}{\partial\lambda^{s_k}}\Delta_k^0(\lambda)\Big)_{|\lambda=0},\tag{23}$$

and $s_k\geqslant 0$ is the multiplicity of the zero eigenvalue. Let us show that

$$\Delta_k(\lambda)=A_k^0\prod_{n=0}^{\infty}\frac{\lambda_{kn}-\lambda}{\lambda_{kn}^{01}}.\tag{24}$$

Indeed, by Hadamard's factorization theorem,

$$\Delta_k(\lambda) = A_k \prod_{n=0}^{\infty} \frac{\lambda_{kn} - \lambda}{\lambda_{kn}^1}, \tag{25}$$

where $A_k \neq 0$ is a constant, $\lambda_{kn}^1 = \lambda_{kn}$ if $\lambda_{kn} \neq 0$, and $\lambda_{kn}^1 = 1$ if $\lambda_{kn} = 0$. It follows from (22) and (25) that

$$\frac{\Delta_k(\lambda)}{\Delta_k^0(\lambda)} = \frac{A_k}{A_k^0} \prod_{n=0}^{\infty} \frac{\lambda_{kn}^{01}}{\lambda_{kn}^1} \prod_{n=0}^{\infty} \Big(1 + \frac{\lambda_{kn} - \lambda_{kn}^0}{\lambda_{kn}^0 - \lambda}\Big).$$

Using properties of the characteristic functions and the eigenvalues of B_k one gets for negative λ:

$$\lim_{\lambda \to -\infty} \frac{\Delta_k(\lambda)}{\Delta_k^0(\lambda)} = 1, \qquad \lim_{\lambda \to -\infty} \prod_{n=0}^{\infty} \Big(1 + \frac{\lambda_{kn} - \lambda_{kn}^0}{\lambda_{kn}^0 - \lambda}\Big) = 1,$$

and consequently,

$$A_k = A_k^0 \prod_{n=0}^{\infty} \frac{\lambda_{kn}^1}{\lambda_{kn}^{01}}.$$

Substituting this relation into (25) we arrive at (24).

Thus, the specification of the spectrum $\Lambda_k = \{\lambda_{kn}\}_{n \geqslant 0}$ uniquely determines the characteristic function $\Delta_k(\lambda)$ by (24), where A_k^0 and $\{\lambda_{kn}^{01}\}$ are defined by (21), (23), (18), and (19).

2. Solution of the inverse problem

In this section we provide an algorithm for the solution of Inverse Problem 1 and prove the uniqueness of the solution. First we consider auxiliary inverse problems.

Fix $k = \overline{1, r}$, and consider the following auxiliary inverse problem on the edge e_k, which is called IP(k).

IP(k). Given $M_k(\lambda)$, construct $q_k(x_k)$, $x_k \in [0, T_k]$ and b_k.

In IP(k) we construct the potential only on the edge e_k, but the Weyl function $M_k(\lambda)$ brings global information from the whole graph. In other words, IP(k) is not a local inverse problem related to the edge e_k. Let us prove the uniqueness theorem for the solution of IP(k).

Theorem 2. *If $M_k(\lambda) = \tilde{M}_k(\lambda)$, then $q_k(x_k) = \tilde{q}_k(x_k)$ a.e. on $[0, T_k]$ and $b_k = \tilde{b}_k$. Thus, the specification of the Weyl function M_k uniquely determines the potential q_k on the edge e_k and b_k.*

Proof. Let us define the matrix $P^k(x_k, \lambda) = [P_{js}^k(x_k, \lambda)]_{j,s=1,2}$ by the formula

$$P^k(x_k, \lambda) \begin{bmatrix} \tilde{\varphi}_{kk}(x_k, \lambda) & \tilde{\Phi}_{kk}(x_k, \lambda) \\ \tilde{\varphi}'_{kk}(x_k, \lambda) & \tilde{\Phi}'_{kk}(x_k, \lambda) \end{bmatrix} = \begin{bmatrix} \varphi_{kk}(x_k, \lambda) & \Phi_{kk}(x_k, \lambda) \\ \varphi'_{kk}(x_k, \lambda) & \Phi'_{kk}(x_k, \lambda) \end{bmatrix}. \tag{26}$$

Then (26) yields

$$\varphi_k(x_k, \lambda) = P_{11}^k(x_k, \lambda) \tilde{\varphi}_k(x_k, \lambda) + P_{12}^k(x_k, \lambda) \tilde{\varphi}'_k(x_k, \lambda). \tag{27}$$

Since $\langle \varphi_k(x_k,\lambda),\, \Phi_{kk}(x_k,\lambda)\rangle \equiv 1$, one has

$$P^k_{1s}(x_k,\lambda) = (-1)^{s-1}\Big(\varphi_k(x_k,\lambda)\tilde\Phi^{(2-s)}_{kk}(x_k,\lambda) - \tilde\varphi^{(2-s)}_k(x_k,\lambda)\Phi_{kk}(x_k,\lambda)\Big). \qquad (28)$$

It follows from (15), (16) and (28) that

$$P^k_{1s}(x_k,\lambda) = \delta_{1s} + O(\rho^{-1}), \quad \rho\in\Lambda^\delta, \quad |\rho|\to\infty, \quad x_k\in(0,T_k]. \qquad (29)$$

According to (4) and (28),

$$P^k_{1s}(x_k,\lambda) = (-1)^{s-1}\Big(\Big(\varphi_k(x_k,\lambda)\tilde S^{(2-s)}_k(x_k,\lambda) - \tilde\varphi^{(2-s)}_k(x_k,\lambda)S_k(x_k,\lambda)\Big) + \\ +(M_k(\lambda)-\tilde M_k(\lambda))\varphi_k(x_k,\lambda)\tilde\varphi^{(2-s)}_k(x_k,\lambda)\Big).$$

Since $M_k(\lambda) = \tilde M_k(\lambda)$, it follows that for each fixed x_k, the functions $P^k_{1s}(x_k,\lambda)$ are entire in λ of order $1/2$. Together with (29) this yields $P^k_{11}(x_k,\lambda)\equiv 1$, $P^k_{12}(x_k,\lambda)\equiv 0$. Substituting these relations into (27) we get $\varphi_k(x_k,\lambda)\equiv\tilde\varphi_k(x_k,\lambda)$ for all x_k and λ, and consequently, $q_k(x_k)=\tilde q_k(x_k)$ a.e. on $[0,T_k]$ and $b_k=\tilde b_k$. Theorem 2 is proved. □

Using the method of spectral mappings [2] for the Sturm – Liouville operator on the edge e_k one can get a constructive procedure for the solution of the inverse problem IP(k).

Consider the following auxiliary inverse problem on the edge e_0, which is called IP(0).

IP(0). Given $d(\lambda)$, $h(\lambda)$, Ω, construct $q_0(x_0)$, $x_0\in[0,T_0]$.

This inverse problem was studied in many papers (see [5] and the references therein). For convenience of the readers we describe here the solution of IP(0). We remind that $d(\lambda) = C_0(T_0,\lambda) + S_0'(T_0,\lambda)$, $H(\lambda) = C_0(T_0,\lambda) - S_0'(T_0,\lambda)$, $\omega_n = \operatorname{sign} H(\nu_n)$, where $\{\nu_n\}_{n\geqslant 1}$ are zeros of $h(\lambda)$. Clearly,

$$S_0'(T_0,\nu_n) = (d(\nu_n) - H(\nu_n))/2. \qquad (30)$$

Since $\langle C_0(T_0,\lambda), S_0(T_0,\lambda)\rangle\equiv 1$, it follows that

$$H^2(\lambda) - d^2(\lambda) = -4(1 + C_0'(T_0,\lambda)h(\lambda)),$$

and, consequently,

$$H(\nu_n) = \omega_n\sqrt{d^2(\nu_n)-4}. \qquad (31)$$

Denote $\alpha_n := \displaystyle\int_0^{T_0} S_0^2(t,\nu_n)\,dt$. Then

$$\alpha_n = \dot h(\nu_n)S_0'(T_0,\nu_n), \quad \dot h(\lambda) := \frac{dh(\lambda)}{d\lambda}. \qquad (32)$$

The data $\{\nu_n,\alpha_n\}_{n\geqslant 1}$ are called the spectral data for the potential q_0. It is known (see [1,2]) that the function q_0 can be uniquely constructed from the given spectral data $\{\nu_n,\alpha_n\}_{n\geqslant 1}$. Thus, IP(0) has been solved, and the following theorem is valid.

Theorem 3. *The specification of $d(\lambda), h(\lambda), \Omega$ uniquely determines the potential $q_0(x_0)$ on $[0,T_0]$. The function q_0 can be constructed by the following algorithm.*

Algorithm 1. Given $d(\lambda)$, $h(\lambda)$, Ω.

1. Find $\{\nu_n\}_{n\geqslant 1}$ as the zeros of $h(\lambda)$.
2. Calculate $H(\nu_n)$ by (31).
3. Find $S_0'(T_0,\nu_n)$ by (30).
4. Calculate $\{\alpha_n\}_{n\geqslant 1}$ using (32).
5. Construct q_0 from the given spectral data $\{\nu_n,\alpha_n\}_{n\geqslant 1}$ by solving the classical inverse Sturm – Liouville problem.

Let us go on to the solution of Inverse Problem 1. First we give the proof of Theorem 1. Assume that $\Lambda_k=\tilde\Lambda_k$, $k=\overline{0,r}$, and $\Omega=\tilde\Omega$. Then $\Delta_k(\lambda)\equiv\tilde\Delta_k(\lambda)$, $\quad k=\overline{0,r}$. By virtue of (9) this yields $M_k(\lambda)\equiv\tilde M_k(\lambda)$, $\quad k=\overline{1,r}$. Applying Theorem 2 for each fixed $k=\overline{1,r}$, we obtain

$$q_k(x_k)=\tilde q_k(x_k)\ \text{ a.e. on }\quad [0,T_k],\quad b_k=\tilde b_k,\quad k=\overline{1,r},$$

and, consequently, $C_k(x_k,\lambda)\equiv\tilde C_k(x_k,\lambda)$, $S_k(x_k,\lambda)\equiv\tilde S_k(x_k,\lambda)$, $k=\overline{1,r}$, $x_k\in[0,T_k]$. Taking (6) and (10) into account we deduce $d(\lambda)=\tilde d(\lambda)$, $\quad h(\lambda)=\tilde h(\lambda)$. Since $\Omega=\tilde\Omega$, it follows from Theorem 3 that $q_0(x_0)=\tilde q_0(x_0)$ a.e. on $[0,T_0]$, and Theorem 1 is proved.

The solution of Inverse Problem 1 can be constructed by the following algorithm.

Algorithm 2. Given Λ_k, $k=\overline{0,r}$ and Ω.

1. Construct $\Delta_k(\lambda)$, $k=\overline{0,r}$ by (24), where A_k^0 and $\{\lambda_{kn}^{01}\}$ are defined by (21), (23), (18), and (19).
2. Find $M_k(\lambda)$, $k=\overline{1,r}$ via (9).
3. For each fixed $k=\overline{1,r}$, solve the inverse problem IP(k) and find $q_k(x_k)$, $x_k\in[0,T_k]$ on the edge e_k and b_k.
4. For each fixed $k=\overline{1,r}$, construct $C_k(x_k,\lambda)$, $S_k(x_k,\lambda)$, $x_k\in[0,T_k]$.
5. Calculate $d(\lambda)$ and $h(\lambda)$ using (6) and (10).
6. Construct $q_0(x_0)$, $x_0\in[0,T_0]$ from $d(\lambda),h(\lambda),\Omega$ using Algorithm 1.

References

1. Freiling G., Yurko V. A. *Inverse Sturm – Liouville Problems and their Applications*. New York, NOVA Science Publishers, 2001. 305 p.
2. Yurko V. A. *Method of Spectral Mappings in the Inverse Problem Theory*. (Inverse and Ill-posed Problems Series, vol. 31). Utrecht, VSP, 2002. 316 p.
3. Yurko V. A. Inverse spectral problems for differential operators on spatial networks. *Russian Mathematical Surveys*, 2016, vol. 71, no. 3, pp. 539–584. http://dx.doi.org/10.1070/RM9709
4. Bellmann R., Cooke K. *Differential-Difference Equations*. New York, Academic Press, 2012. 482 p.
5. Yurko V. A. Inverse problems for differential operators with nonseparated boundary conditions in the central symmetric case. *Tamkang Journal of Mathematics*, 2017, vol. 48, no. 4, pp. 377–387. https://doi.org/10.5556/j.tkjm.48.2017.2492

Поступила в редакцию / Received 26.01.2021
Принята к публикации / Accepted 14.03.2021
Опубликована / Published 31.08.2021