Известия Саратовского университета. Новая серия. Серия: Математика. Механика. Информатика. 2022. Т. 22, вып. 2. С. 152–158
Izvestiya of Saratov University. Mathematics. Mechanics. Informatics, 2022, vol. 22, iss. 2, pp. 152–158
https://mmi.sgu.ru https://doi.org/10.18500/1816-9791-2022-22-2-152-158

Article

Stochastic model of innovation diffusion that takes into account the changes in the total market volume

A. Yu. Parphenova, L. A. Saraev✉

Samara National Research University, 34 Moskovskoye shosse, Samara 443086, Russia

Alena Yu. Parphenova, egorovaalena@inbox.ru, https://orcid.org/0000-0001-7374-3663, AuthorID: 783145

Leonid A. Saraev, saraev.leo@gmail.com, https://orcid.org/0000-0003-3625-5921, AuthorID: 2919

Abstract. The article proposes a stochastic mathematical model of the diffusion of consumer innovations, which takes into account changes over time in the total number of potential buyers of an innovative product. A stochastic differential equation is constructed for a random value of the number of consumers of an innovative product. The interaction of random changes in the number of consumers with changes in the total market volume of the product under consideration is investigated. Following the Euler – Maruyama method, an algorithm for the numerical solution of the stochastic differential equation for the diffusion of innovations is constructed. For each implementation of this algorithm, the corresponding stochastic trajectories are constructed for a random function of the number of consumers of an innovative product. A variant of the method for calculating the mathematical expectation of a random function of the number of consumers of an innovative product is developed and the corresponding differential equation is obtained. It is shown that the numerical solution of this equation and the average value of the function of the number of consumers calculated for all the implemented implementations of stochastic trajectories give practically the same results. Numerical analysis of the developed model showed that taking into account an external random disturbing factor in the stochastic model leads to significant deviations from the classical deterministic model of smooth market development with innovative goods.

Keywords: innovation diffusion, stochastic equations, Wiener process, innovation coefficient, simulation coefficient

For citation: Parphenova A. Yu., Saraev L. A. Stochastic model of innovation diffusion that takes into account the changes in the total market volume. *Izvestiya of Saratov University. Mathematics. Mechanics. Informatics*, 2022, vol. 22, iss. 2, pp. 152–158. https://doi.org/10.18500/1816-9791-2022-22-2-152-158

Научная статья
УДК 330.42

Стохастическая модель диффузии инноваций, учитывающая изменение общего объема рынка

А. Ю. Парфенова, Л. А. Сараев✉

Самарский национальный исследовательский университет имени академика С. П. Королева, Россия, 443086, г. Самара, ул. Московское шоссе, д. 34

Парфенова Алена Юрьевна, старший преподаватель кафедры математики и бизнес-информатики, egorovaalena@inbox.ru, https://orcid.org/0000-0001-7374-3663, AuthorID: 783145

Сараев Леонид Александрович, доктор физико-математических наук, заведующий кафедрой математики и бизнес-информатики, saraev.leo@gmail.com, https://orcid.org/0000-0003-3625-5921, AuthorID: 2919

Аннотация. В статье предложена стохастическая математическая модель диффузии потребительских инноваций, учитывающая изменения во времени общего числа потенциальных покупателей инновационного товара. Построено стохастическое дифференциальное уравнение для случайной величины числа потребителей инновационного товара. Исследовано влияние случайных изменений числа потребителей на изменение общего объема рынка рассматриваемого товара. В соответствии с методом Эйлера – Маруямы построен алгоритм численного решения стохастического дифференциального уравнения диффузии инноваций. Для каждой реализации этого алгоритма строятся соответствующие стохастические траектории для случайной функции числа потребителей инновационного товара. Разработан вариант метода расчета математического ожидания случайной функции числа потребителей инновационного товара и получено соответствующее для него дифференциальное уравнение. Показано, что численное решение этого уравнения и среднее значение функции числа потребителей, вычисленное по всем реализациям стохастических траекторий, дают практически одинаковые результаты. Численный анализ разработанной модели показал, что учет в стохастической модели внешнего случайного возмущающего фактора приводит к существенным отклонениям от классической детерминированной модели плавного наполнения рынка инновационными товарами.

Ключевые слова: диффузия инноваций, стохастические уравнения, винеровский процесс, коэффициент инновации, коэффициент имитации

Для цитирования: *Parphenova A. Yu., Saraev L. A.* Stochastic model of innovation diffusion that takes into account the changes in the total market volume [*Парфенова А. Ю., Сараев Л. А.* Стохастическая модель диффузии инноваций, учитывающая изменение общего объема рынка] // Известия Саратовского университета. Новая серия. Серия: Математика. Механика. Информатика. 2022. Т. 22, вып. 2. С. 152–158. https://doi.org/10.18500/1816-9791-2022-22-2-152-158

Introduction

One of the important and relevant areas of modern economic theory is the development of economic and mathematical methods for predicting the features and patterns of stochastic processes of distribution in the markets of goods with new properties.

New stochastic models developed using these methods to assess the parameters of the diffusion of consumer innovations help to assess the growth rate of sales of goods with new properties, adequately predict the indicators of market capture by them, calculate the time intervals of stagnation or decline in their sales, etc.

A significant contribution to the development of theoretical principles of the diffusion of consumer innovations is presented in the works [1–4].

On the basis of these provisions, researchers have developed a number of models for assessing the parameters of the diffusion of innovations, in which the total market size of the considered innovative product was considered constant [5–8].

The nature of the dynamic development of the process of diffusion of consumer innovations is determined by the ratio of consumers-innovators and consumers-imitators, the volatility of the volume of sales and changes in the total volume of the market, consisting of the total number of potential buyers of the innovative product.

Many well-known statistics on sales of various innovative products show the stochastic nature of their distribution among consumers. Therefore, while constructing mathematical models of the dynamics of diffusion of innovations we should rely on the theory of random functions. Stochastic modeling makes it possible to fully take into account the unstable nature of the market and make significant additions to the existing similar deterministic models.

An effective tool for constructing non-deterministic models of diffusion of innovations is the theory of stochastic differential equations, which takes into account the influence of random external influences. The construction of the defining equations for the diffusion of innovations based on this theory significantly enriches the corresponding well-known deterministic models, in which external random disturbing factors cannot be taken into account. Methods of studying applications of the stochastic differential equations theory for modeling random processes are described in detail in [9–13].

This work aims to develop a new economic and mathematical model of innovation diffusion, taking into account the random nature of consumer behavior and possible variations in the total market volume.

The peculiarity of the proposed model and its scientific novelty lies in the fact that, in contrast to the classical models of innovation diffusion, it takes into account the change in the total volume of the market of potential buyers over time.

1. Stochastic differential equation of innovation diffusion in an unstable market

Let some fundamentally new innovative product appear and spread on the market. Let us denote $U(t)$ as the volume of that part of the market, which is formed by all purchasers of this product at the moment of time t, $V(t)$ as the maximum volume of that part of the market, which is formed by all potential purchasers of this product at the moment time t. The units of measurement for the values of $U(t)$ and $V(t)$ can be percent — the number of buyers of the product and the relative number of buyers of the product. In the latter case, the maximum market volume for the entire period under consideration is taken as a unit. The function $U(t)$ is assumed to be continuous and continuously differentiable on the interval $(0 \leqslant t < \infty)$.

The increment in the number of buyers of the innovative product ΔU over a certain period of time Δt can be represented in the form of three terms

$$\Delta U = \Delta U^N + \Delta U^I + \Delta U^W, \tag{1}$$

where ΔU^N is a partial increment over the time interval Δt of the number of innovative buyers oriented towards advertising and the media, ΔU^I is a partial increment over the interval time Δt of the number of imitator buyers relying on reviews of people who have already made a purchase, ΔU^W stands for random fluctuations in the number of imitator buyers due to market volatility.

The quantities ΔU^N, ΔU^I, ΔU^W can be represented as [14]

$$\begin{cases} \Delta U^N(t) = a \cdot V(t) \cdot \left(1 - \dfrac{U(t)}{V(t)}\right) \cdot \Delta t, \\ \Delta U^I(t) = b \cdot U(t) \cdot \left(1 - \dfrac{U(t)}{V(t)}\right) \cdot \Delta t, \\ \Delta U^W(t) = \rho \cdot b \cdot U(t) \cdot \left(1 - \dfrac{U(t)}{V(t)}\right) \cdot \Delta w, \end{cases} \tag{2}$$

where a is the coefficient of innovation, which determines the share of buyers-innovators from the total number of potential buyers $V(t)$, b is the coefficient of imitation, which determines the share of buyers-imitators from the number of buyers who have already made a purchase $U(t)$, w is standard Wiener process, $\Delta w = \epsilon \cdot \sqrt{\Delta t}$, ρ is market volatility, ε is a random variable with normal distribution, zero mean $\langle \varepsilon \rangle = 0$, and unit variance $\langle \varepsilon^2 \rangle = 1$.

The multiplier $\left(1 - \dfrac{U(t)}{V(t)}\right)$ describes the process of market saturation up to a certain limit value $V(t)$.

It should be noted that in the considered model, in contrast to the classical model by F. Bass, the total number of potential buyers $V(t)$ is not a constant but is assumed to be a variable [14].

Substitution of the quantities (2) into the equation (1) leads to the relation

$$\Delta U(t) = \left(1 - \frac{U(t)}{V(t)}\right) \cdot \left(\Big(a \cdot V(t) + b \cdot U(t)\Big) \cdot \Delta t + \rho \cdot b \cdot U(t) \cdot \Delta w\right). \tag{3}$$

Passing to the limit in the formula (3) for $\Delta t \to 0$, $\Delta w \to 0$ allows one to obtain the nonlinear stochastic Ito differential equation [12]

$$dU(t) = S\big(U(t), t\big) \cdot \Delta t + Z\big(U(t), t\big) \cdot \Delta w, \tag{4}$$

where

$$S\big(U(t), t\big) = \Big(a \cdot V(t) + b \cdot U(t)\Big) \cdot \left(1 - \frac{U(t)}{V(t)}\right) \tag{5}$$

is a drift coefficient,

$$Z\big(U(t), t\big) = \rho \cdot b \cdot U(t) \cdot \left(1 - \frac{U(t)}{V(t)}\right) \tag{6}$$

is a volatility coefficient.

The initial condition for the equation (4) has the form

$$U(0) = U_0. \tag{7}$$

It is obvious that if the process of diffusion of innovations is observed from the very beginning, then $U_0 = 0$. Otherwise, the value of U_0 may differ from zero.

2. Diffusion model of innovation for a market with the variable total volume

Let us consider a variant of the development of promoting an innovative product to the market, in which at the initial moment of time t the maximum number of its potential buyers is V_0, and then this number gradually decreases to its limiting value V_∞. In this case, the function of the total market volume $V(t)$ can be represented as [14]

$$V(t) = V_\infty + (V_0 - V_\infty) \cdot \exp(-\lambda \cdot t), \tag{8}$$

where λ is a parameter characterizing the rate of change in the total number of potential buyers of an innovative product.

The numerical solution of the equation (4) with the coefficients (5), (6) and the initial condition (7) is performed on the divided system of points $t_0 < t_1 < t_2 < ... < t_n$ time interval $[t_0, t_n]$ by the method of successive approximations of Euler – Maruyama in accordance with the algorithm [3]

$$U_{s+1} = U_s + S(U_s, t_s) \cdot \Delta t_s + \varepsilon_s \cdot Z(U_s, t_s) \cdot \sqrt{\Delta t_s}\,, \ (s = 0, 1, 2, ..., n-1), \tag{9}$$

where n is the number of time partitions.

At each small time step $\Delta t_s = t_{s+1} - t_s$, starting from the initial value U_0, a random number ε_s is generated and the following is calculated the value of U_{s+1}.

Thus, random sequences $\{t_s\}$ and $\{U_s\}$ are formed. On the coordinate plane, these sequences form a system of points $\{t_s, U_s\}$ and the corresponding stochastic trajectory.

When repeating the implementation of the algorithm (9) m times, each time a new stochastic trajectory is formed, since each time the random variable ε generates new random values.

To find the mathematical expectation of the function $\langle U(t)\rangle$, it is necessary to statistically average the equation (4) with the coefficients (5) and (6)

$$\Big\langle dU(t) \Big\rangle = \left\langle \Big(a \cdot V(t) + b \cdot U(t)\Big) \cdot \left(1 - \frac{U(t)}{V(t)}\right) \right\rangle \cdot \Delta t. \tag{10}$$

The result is a differential equation containing the statistical moment of the required second-order function $\langle U(t)^2\rangle$

$$\frac{d\langle U(t)\rangle}{dt} = V(t) \cdot \left(a + (b-a) \cdot \frac{\langle U(t)\rangle}{V(t)} - b \cdot \frac{\langle U(t)^2\rangle}{V(t)^2}\right). \tag{11}$$

Sequential calculation of the moment $\langle U(t)^2\rangle$ leads to the appearance of moments of the third, fourth and higher orders. An endless chain of statistical equations is formed, which must be cut off by making certain assumptions.

In the case under consideration, it is natural to assume that fluctuations in the value of $U(t)$ are determined by random fluctuations in the number of buyers-imitators, and it can be represented as [15]

$$U(t) = \langle U(t)\rangle + \rho \cdot \langle U(t)\rangle \cdot \left(1 - \frac{\langle U(t)\rangle}{V(t)}\right) \cdot \varepsilon. \tag{12}$$

Then

$$U(t)^2 = \langle U(t)\rangle^2 \cdot \left(1 + 2 \cdot \rho \cdot \left(1 - \frac{\langle U(t)\rangle}{V(t)}\right) \cdot \varepsilon + \rho^2 \cdot \left(1 - \frac{\langle U(t)\rangle}{V(t)}\right)^2 \cdot \varepsilon^2\right). \tag{13}$$

Averaging the equality (13) gives

$$\langle U^2\rangle = \langle U\rangle^2 \cdot \left(1+\rho^2\cdot\left(1-\frac{\langle U\rangle}{V}\right)^2\right). \tag{14}$$

Substituting the formula (14) into the ratio (11), we find the differential equation for the mathematical expectation of the value $U(t)$

$$\frac{d\langle U\rangle}{dt} = V\cdot\left(a+(b-a)\cdot\frac{\langle U\rangle}{V} - b\cdot\frac{\langle U\rangle^2}{V^2}\cdot\left(1+\rho^2\cdot\left(1-\frac{\langle U\rangle}{V}\right)^2\right)\right). \tag{15}$$

The initial condition for equation (15) has the form

$$\langle U(0)\rangle = U_0. \tag{16}$$

The Figure shows five from a hundred stochastic trajectories obtained as a result of numerical implementations of the Euler – Maruyama algorithm (9), (7), numerical solution of the Cauchy problem (15), (16), as well as graphs of the function of the total market volume $V(t)$ and its limit value V_∞.

When comparing the numerical solution of the Cauchy problem (15), (16) for the mathematical expectation of the function $U(t)$ and the mean value of this function calculated for all $m = 100$ implementations of the algorithm (9) it turned out that they practically coincide.

For a deterministic process of diffusion of innovations at $\rho = 0$, the results obtained coincide with those of [14].

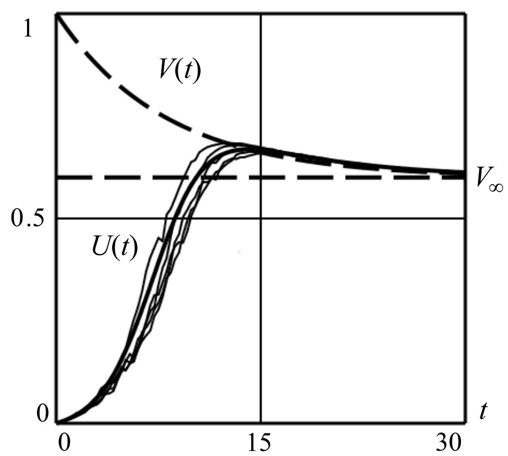

Figure. Stochastic trajectories constructed from the results of numerical implementations of the Euler – Maruyama algorithm (9), (7) and the expectation curve for the function $U(t)$, constructed in accordance with the numerical solution of the Cauchy problem (15), (16) (solid lines). Graph of the total market volume function $V(t)$ and its asymptote $V = V_\infty$ (dashed lines). Calculated values: $t_0 = 0$; $t_n = 30$; $n = 100$; $m = 100$; $V_0 = 1$; $V_\infty = 0,6$; $a = 0,01$; $b = 0,5$; $\rho = 0,35$.

3. Conclusion

A new model of innovation diffusion has been developed in the form of a stochastic differential equation that takes into account the change in the total number of potential buyers over time.

An algorithm for the numerical solution of the obtained stochastic differential equation for constructing stochastic trajectories of a random function of the number of consumers of an innovative product is presented.

A variant of the method of statistical averaging of the stochastic differential equation for the diffusion of innovations has been developed and a differential equation has been obtained to determine the mathematical expectation of a random function of the number of consumers of an innovative product.

It was found that the numerical solution of this equation and the statistical average value of the function of the number of consumers of an innovative product, calculated for all realizations of stochastic trajectories, give almost the same results.

References

1. Bass F. M. A new product growth model for consumer durables. *Management Science*, 1969, vol. 15, no. 5, pp. 215–227. https://doi.org/10.1287/mnsc.1040.0264
2. Bass F. M. The future of research in marketing: Marketing science. *Journal of Marketing Research*, 1993, vol. 30, pp. 1–6. https://doi.org/10.1177/002224379303000101
3. Bass F. M. Empirical generalizations and marketing science: A personal view. *Marketing Science*, 1995, vol. 14, pp. 6–19. https://doi.org/10.1287/mksc.14.3.G6
4. Rogers E. M. *Diffusion of Innovations*. 5th ed. New York, Free Press, 2003. 576 p.
5. Srinivasan V., Mason C. H. Nonlinear least squares estimation of new product diffusion model. *Marketing Science*, 1986, vol. 5, iss. 2, pp. 169–178.
6. Schmittlein D. C., Mahajan V. Maximum likelihood estimation for an innovational diffusion model of new-product acceptance. *Management Science*, 1982, vol. 1, no. 1, pp. 57–78. https://doi.org/10.1287/MKSC.1.1.57
7. Mahajan V., Sharma S. A simple algebraic estimation procedure for innovation diffusion models of new product acceptance. *Technological Forecasting and Social Change*, 1986, vol. 30, iss. 4, pp. 331–345. https://doi.org/10.1016/0040-1625(86)90031-4
8. Mahajan V., Mason C. H., Srinivasan V. An evaluation of estimation procedures for new product diffusion models. In: V. Mahajan, Y. Wind, eds. *Innovation Diffusion Models of New Product Acceptance*. Cambridge, Ballinger Publishing Company, 1986, pp. 203–232.
9. Itô K., McKean H. P. *Diffusion Processes and their Sample Paths*. Berlin, Heidelberg, Springer, 1996. 323 p. https://doi.org/10.1007/978-3-642-62025-6 (Russ. ed.: Moscow, Mir, 1986. 329 p.).
10. Allen E. *Modeling with Itô Stochastic Differential Equations*. Mathematical Modelling: Theory and Applications, vol. 22. Springer, 2007. 230 p. https://doi.org/10.1007/978-1-4020-5953-7
11. Kanellos N., Katsianis D., Varoutas D. On the Introduction of Diffusion Uncertainty in Telecommunications' Market Forecasting. *Engineering Proceedings*, 2021, vol. 5, no. 1, Art. 13. https://doi.org/10.3390/engproc2021005013
12. Wang H., Sun B. Diffusion Mechanism of Leading Technology in the New Energy Industry Based on the Bass Model. *Frontiers in Energy Research*, 2021, vol. 9. https://doi.org/10.3389/fenrg.2021.586787
13. Grasman J., Kornelis M. Forecasting product sales with a stochastic Bass model. *Journal of Mathematics in Industry*, 2019, vol. 9, Art. 2. https://doi.org/10.1186/s13362-019-0059-6
14. Ilyina E. A., Parphenova A. Yu., Saraev L. A. Influence of changes to the total volume of the market on the kinetics of the process of diffusion of innovations. *Vestnik Altaiskoi akademii ekonomiki i prava* [Bulletin of the Altai Academy of Economics and Law], 2019, no. 12, pp. 61–67 (in Russian). https://doi.org/10.17513/vaael.848
15. Saraev A. L., Saraev L. A. Stochastic calculation of curves dynamics of enterprise. *Journal of Samara State Technical University, Ser. Physical and Mathematical Sciences*, 2020, vol. 24, no. 2, pp. 343–364 (in Russian). https://doi.org/10.14498/vsgtu1700

Поступила в редакцию / Received 21.12.2021
Принята к публикации / Accepted 18.01.2022
Опубликована / Published 31.05.2022