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Article

Generalized pseudotensor formulations of the Stokes' integral theorem

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Abstract. Oriented continua play an important role in micropolar elasticity modelling. All realizations of micropolar theories are conceptually possible only within the framework of the pseudotensor formalism and the orientable manifold notion. This particularly concerns the theory of micropolar hemitropic elastic media. In this paper, a pseudotensor description is used in contrast to Kartan's formalism. The pseudotensor formulation of Stokes' integral theorem is almost unknown in the current scientific literature. Here we consider various formulations of Stokes' integral theorem for an arbitrary asymmetric covariant pseudotensor field of a given weight and valency. This extends the theorem to the case of pseudotensors. This fact makes it possible to use the mentioned generalization for micropolar continua. The study mostly relies on the class of special coordinate systems often employed in classical physical field theories. A procedure for orientations consistency inside and on the boundary of a manifold is discussed for various formulations of Stokes' integral theorem.

Keywords: pseudotensor, fundamental orienting pseudoscalar, micropolar hemitropic continuum, M-cell, coordinate frame, Stokes' integral theorem, orientation consistency
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Обобщенные псевдотензорные формулировки интегральной теоремы Стокса

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Аннотация. Ориентируемые континуумы играют важную роль в микрополярной теории упругости, все реализации которой возможны только в рамках псевдотензорного формализма и представления об ориентируемом многообразии. Особенно это касается теории микрополярных гемитропных упругих сред. В настоящей статье используется псевдотензорное описание, а не формализм Картана. В литературе неизвестна псевдотензорная формулировка теоремы Стокса. Рассматриваются различные формулировки интегральной теоремы Стокса для асимметричного ковариантного псевдотензорного поля, заданного веса и валентности. Тем самым достигается распространение известной интегральной формулы Стокса на случай псевдотензоров. Последнее обстоятельство позволяет использовать указанное обобщение для микрополярных континуумов. Исследование существенно опирается на класс специальных координатных систем. Обсуждается процедура согласования ориентаций реперов внутри и на границе многообразия для различных формулировок интегральной теоремы Стокса.

Ключевые слова: псевдотензор, фундаментальный ориентирующий псевдоскаляр, микрополярный гемитропный континуум, M-ячейка, репер, интегральная теорема Стокса, согласование ориентаций

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Preliminary remarks

An orientable manifold [1–3] naturally appears in micropolar elasticity. In continuum mechanics, it is usually required that the continuum be immersed in an external "plane" space. This is of crucial importance for modeling the mechanical behaviour of materials with microstructure, micropolar media [4], and industrial processes of additive manufacturing [5, 6]. Special attention should be paid to the orientations consistency of elementary cell repers frames taken inside the continuum and on its boundary. It is also true for integral theorems and balance laws of non-linear continuum mechanics. The tensor elementary volume measurements lead to the employment of pseudotensor calculus and the fundamental concept of orientable manifolds [1–3, 7].

Throughout the paper, we are going to follow the terminology and notions related to multidimensional geometry presented in [3]. The requisite notations and equations for the tensor elementary volume and area can be found in monographs [1, see Appendix on Invariants by J. L. Ericksen] and [3]. Applying pseudotensor algebra to the problems of mechanics of growing solids and micropolar elasticity are discussed in the studies [5–10].

In the present paper, the problems of orientations consistency (situated inside and on the boundary) for a micropolar continuum immersed in an external plane space of a given dimension N are discussed.

After the Preliminary remarks in Sec. 1, the definitions of fundamental orienting pseudoscalar, covariant derivative, M-manifold, M-cell, and tensor elementary volume are recalled for N-dimensional space. The covariant differentiation of an arbitrary relative tensor is considered. Based on the concept of an M-cell tensor elementary volume (area), an algorithm for comparing and spatial orientations consistency of M-cells is described. Processes of continuous transfer of directions associated with an M-cell along a given path are considered.

Then, in Sec. 2 generalized pseudotensor formulations of Stokes' integral theorem are refined due to the class of special coordinate nets geometrically constrained by the equation $e^2 = \overset{[2]}{1}$[1].

Finally, in Sec. 3 the orientations consistency procedure for Stokes' integral theorem formulations in the case of a 2-manifold is considered and discussed. The preferable bypasses of boundary contours associated with boundary 1-cell direction for different formulations of Stokes' integral theorem are discriminated and discussed.

As a whole, the present paper should be considered as a contemporary framework for problems of Stokes' integral theorem formulations for tensors and pseudotensors fields that is important for non-linear continuum mechanics.

1. Orientation of M-manifold. Pseudotensors of integer weights. Tensor and pseudotensor elementary volume

In this study we shall not recall the definitions and properties of pseudotensors. A detailed discussion of pseudotensor algebra and analysis can be found in the tensor analysis textbooks [2, 3] and in the papers [5–10]. Hereafter, the weight of a relative tensor (pseudotensor) is embraced by square brackets situated above the root symbol. The zero weight of absolute tensors will not be noted.

Consider an N-dimensional Euclidean "plane" space supplied by a curvilinear coordinate net x^k $(k = 1, 2, \dots, N)$, local covariant basis $\underset{1}{\boldsymbol{\imath}}, \underset{2}{\boldsymbol{\imath}}, \dots, \underset{N}{\boldsymbol{\imath}}$ and the metric tensor g_{ij}. A manifold (surface) of mathematical dimension M $(M \leqslant N)$ immersed in external space is called an M-manifold. Consider two repers with different corner points x^k and $\overline{x}^k$ and endpoints $x^k + \underset{\mathfrak{c}}{dx^k}$ and $\overline{x}^k + \underset{\mathfrak{c}}{\overline{dx}^k}$ $(\mathfrak{c} = 1, 2, \dots, M)$. Then the external coordinates of the first and the second reper vectors are $\underset{\mathfrak{c}}{dx^k}$ and $\underset{\mathfrak{c}}{\overline{dx}^k}$ respectively.

Orientable manifolds are of crucial importance for the micropolar theories of continuum mechanics [4]. It is then clear that the reper orientation at a micropolar solid point is determined by an enumeration of directions. The reper orientation is changed after a permutation of two numbers of reper directions, i.e., the right-handed frame becomes left-handed. In continuum mechanics, the orientation of the coordinate frame is conveniently determined by the sign of fundamental orienting pseudoscalar e [5, 8]. In three-dimensional space, e can be defined by the triple product of the covariant basis vectors

$$e = \overset{[+1]}{e} = (\underset{1}{\boldsymbol{\imath}} \times \underset{2}{\boldsymbol{\imath}}) \cdot \underset{3}{\boldsymbol{\imath}}. \tag{1}$$

Evidently, the orienting pseudoscalar (1) apart of the sign is the parallelepiped volume constituted by the vectors $\underset{\mathfrak{a}}{\boldsymbol{\imath}}$. It is easy to show that

$$e^2 = g, \quad g = \det(g_{ij}). \tag{2}$$

[1]The terminology and notations worked out in [7, 9, 10] are employed throughout the paper.

It is clear that $e \overset{[+1]}{>0}$ for a right-handed coordinate net, and $e \overset{[+1]}{<0}$ for a left-handed coordinate net.

Note that the fundamental orienting pseudoscalar allows us to transform a pseudotensor of a given weight W and valency into an absolute tensor following the rule

$$\mathbf{T} = e^{-W} \overset{[W]}{\mathbf{T}}. \qquad (3)$$

By comparing weights on the left and right sides in (3), we conclude that $\mathbf{T}$ is an absolute tensor. In what follows, for fundamental symbols such as e, the indication of their weights is usually omitted.

The covariant derivative of a pseudotensor $T^{lm\cdots n}_{ij\cdots k}$ of a given weight W is defined similarly to the corresponding operation for absolute tensors [2, 3, 8]:

$$\nabla_p \overset{[W]}{T}{}^{lm\ldots n}_{ij\ldots k} = \partial_p \overset{[W]}{T}{}^{lm\ldots n}_{ij\ldots k} + \overset{[W]}{T}{}^{sm\ldots n}_{ij\ldots k}\Gamma^l_{sp} + \cdots + \overset{[W]}{T}{}^{lm\ldots s}_{ij\ldots k}\Gamma^s_{ip} -$$
$$-\Gamma^l_{sp}\overset{[W]}{T}{}^{lm\ldots n}_{sj\ldots k} - \cdots - \Gamma^l_{sp}\overset{[W]}{T}{}^{lm\ldots n}_{ij\ldots s} - W\overset{[W]}{T}{}^{lm\ldots n}_{ij\ldots k}\Gamma^s_{sp}.$$

In particular, for a pseudoscalar, the covariant derivative takes the form

$$\nabla_p \overset{[W]}{T} = \partial_p \overset{[W]}{T} - W \overset{[W]}{T} \Gamma^s_{sp}, \qquad (4)$$

wherein

$$\Gamma^s_{sp} = \frac{\partial_p e}{e}.$$

Taking account of the Christoffel symbol properties and equation (2), we obtain an expression for the covariant derivative (4) as follows

$$\nabla_p \overset{[W]}{T} = \partial_p \overset{[W]}{T} - e^{-1} W \overset{[W]}{T} \partial_p e.$$

Let the differentiable M-manifold be parameterized by Gaussian (intrinsic) coordinates u^α $(\alpha = 1, 2, \ldots, M)$:

$$x^k = x^k(u^1, u^2, \ldots, u^M) \quad (k = 1, 2, \ldots, N). \qquad (5)$$

In equation (5) x^k are the external coordinates for the M-manifold and u^α are the intrinsic ones.

Let us divide the M-manifold into a system of M-cells. Each M-cell is defined by a corner reper characterized by a corner point (with external coordinates x^k and intrinsic coordinates u^α) and endpoints of the reper with intrinsic coordinates

$$u^\alpha + \underset{\mathfrak{c}}{d}u^\alpha \quad (\alpha = 1, 2, \ldots, M)$$

or external coordinates

$$x^k + \underset{\mathfrak{c}}{d}x^k \quad (k = 1, 2, \ldots, N),$$

where the “fraktur” index $\mathfrak{c}$ enumerates the reper directions ($\mathfrak{c} = 1, 2, \ldots, M$). From the external (spatial) point of view, the reper directions are given by the absolute contravariant vectors

$$\underset{1}{d}x^k, \underset{2}{d}x^k, \ldots, \underset{M}{d}x^k \quad (k = 1, 2, \ldots, N).$$

We proceed to define the operation of continuous transfer of the M-cell reper along the path Π from the point x^k to the point $\overline{x}^k$ (Fig. 1). Consider a piecewise smooth path Π on the M-manifold linked two points x^k and $\overline{x}^k$. There always exists a linear transformation from one reper to another, acting according to the well-known formula

$$\overline{\underset{\mathfrak{c}}{d}x}^k = P^{\mathfrak{a}\cdot}_{\cdot\mathfrak{c}} \underset{\mathfrak{a}}{d}x^k, \qquad (6)$$

where $P^{\mathfrak{a}\cdot}_{\cdot\mathfrak{c}}$ is the transfer matrix.

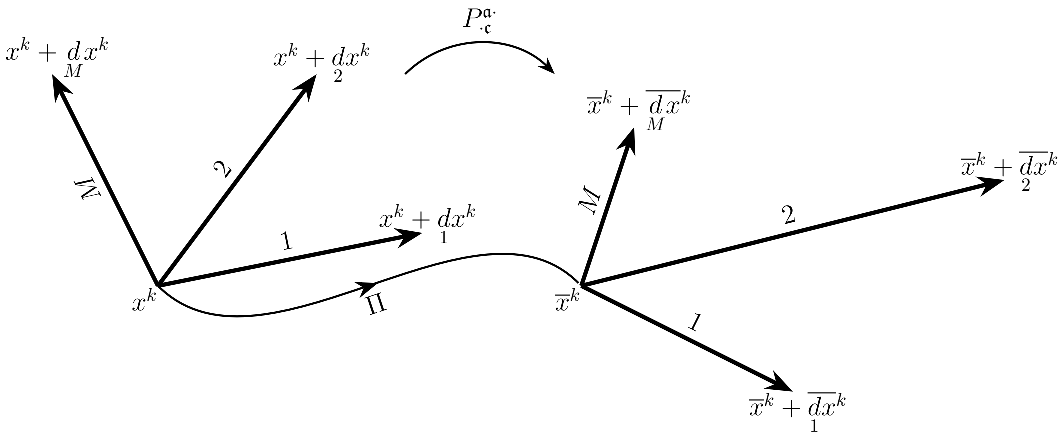

Fig. 1. Directions associated with M-cells. The comparison of orientations is due to continuous transfer along the path Π

The transfer is continuously deforming the reper from the starting point to the endpoint so that the reper directions remain linearly independent. It is assumed that the reper does not degenerate during the transfer process, i.e. the skew product of the reper vectors is never equal to zero along the path Π. The initial and final repers orientations can be then compared: if the orientation of the final reper is the same, i.e. the determinant of the transfer matrix in (6) is positive, then they are treated as co-oriented. If it is true for each pair of repers on M-manifold then the manifold is called orientable or two-sided. Contrariwise, if there exists a pair of differently oriented repers then such a M-manifold is called a non-orientable or one-sided one.

The tensor elementary volume of the M-cell is defined by

$$d\tau^{i_1 i_2 \ldots i_M} = M! \underset{1}{d}x^{[i_1} \underset{2}{d}x^{i_2} \cdots \underset{M}{d}x^{i_M]}. \qquad (7)$$

Here the square brackets denote index alternation.

By considering the following formula for the differentials of external coordinates along with the reper directions of the M-cell

$$\underset{\mathfrak{b}}{d}x^k = (\partial_\alpha x^k) \underset{\mathfrak{b}}{d}u^\alpha,$$

equation (7) reads (see [3, pp. 256–257])

$$d\tau^{i_1 i_2 \ldots i_M} = \epsilon^{\alpha_1 \alpha_2 \ldots \alpha_M} \partial_{\alpha_1} x^{i_1} \partial_{\alpha_2} x^{i_2} \cdots \partial_{\alpha_M} x^{i_M} \det(\underset{\mathfrak{b}}{d}u^\gamma). \qquad (8)$$

The factor before the determinant in (8) is called an intrinsic elementary volume of the M-cell. Using the concept of the intrinsic volume of the M-cell, one can propose an algorithm that orients M-manifold (for details see [3]) if it is possible at all in the sense discussed above.

If elementary M-cells are divided by the coordinate surfaces $u^\alpha = c^\alpha$, then for the case $M = N$ we obtain

$$d\tau^{i_1 i_2 \ldots i_N} = \overset{[-1]}{d\tau}{}^{12\ldots N} \epsilon^{i_1 i_2 \ldots i_N},$$

where $\overset{[-1]}{d\tau}{}^{12\ldots N}$ is the natural elementary volume being a pseudoscalar of weight -1 and defined as follows

$$\overset{[-1]}{d\tau}{}^{12\ldots N} = \det(\partial_\alpha x^k) du^1 du^2 \cdots du^N = dx^1 dx^2 \cdots dx^N.$$

2. A generalized pseudotensor formulation of Stokes' integral theorem

The integral transformation theorems play an important role in continuum mechanics and thermomechanics while formulating conservation laws. For a differentiable asymmetric covariant field $A_{i_1 i_2 \ldots i_{M-1}}$ one can formulate Stokes' integral theorem in the form [3, p. 269]

$$\int \partial_{i_M} A_{i_1 i_2 \ldots i_{M-1}} d\tau^{i_1 i_2 \ldots i_M} = \oint\limits_{\partial} A_{i_1 i_2 \ldots i_{M-1}} d\tau^{i_1 i_2 \ldots i_{M-1}}, \tag{9}$$

where $d\tau^{i_1 i_2 \ldots i_M}$ is the tensor elementary volume of M-manifold calculated due to the intrinsic parameterization u^1, u^2, $\ldots$, u^M whereas $d\tau^{i_1 i_2 \ldots i_{M-1}}$ is the tensor elementary volume of the boundary of M-manifold calculated by an intrinsic parameterization $\widetilde{u}^1, \widetilde{u}^2, \ldots, \widetilde{u}^{M-1}$. The operation with formula (9) implies the orientations consistency inside M-manifold and on the boundary. At this aim, we introduce a non-degenerated vector field s^k on the boundary directed outward with respect to the M-manifold. Arrange the parameterizations u^1, u^2, $\ldots$, u^M and $\widetilde{u}^1, \widetilde{u}^2, \ldots, \widetilde{u}^{M-1}$ so that the corresponding systems of vectors $\partial_1 x^k$, $\partial_2 x^k$, $\ldots$, $\partial_M x^k$ ($\partial_i = \partial/\partial u^i$) and $\widetilde{\partial}_1 x^k$, $\widetilde{\partial}_2 x^k$, $\ldots$, $\widetilde{\partial}_{M-1} x^k$, s^k ($\widetilde{\partial}_i = \partial/\partial \widetilde{u}^i$) are co-oriented in the sense described in Sec. 1.

In previous studies [1, 2] Stokes' integral theorem is formulated as [2, p. 103]

$$\int \partial_{i_1} A_{i_2 i_3 \ldots i_M} d\tau^{i_1 i_2 \ldots i_M} = \oint\limits_{\partial} A_{i_2 i_3 \ldots i_M} d\tau^{i_2 i_3 \ldots i_M}. \tag{10}$$

Once again, arrange the parametrizations inside the M-manifold and on the boundary. In this case, the orientations consistency procedure differs from the one mentioned above by the order of vectors on the boundary of M-manifold. In such a case the vector s^k should be placed in the first position.

Note that the Stokes' theorem formulations (9) and (10) imply that the field $A_{i_1 i_2 \ldots i_{M-1}}$ may not have a tensor nature.

Stokes' theorem (9) remains valid for a pseudotensor field of a given integer weight W

$$\int \partial_{i_M} \overset{[W]}{A}_{i_1 i_2 \ldots i_{M-1}} d\tau^{i_1 i_2 \ldots i_M} = \oint\limits_{\partial} \overset{[W]}{A}_{i_1 i_2 \ldots i_{M-1}} d\tau^{i_1 i_2 \ldots i_{M-1}}. \tag{11}$$

Invariant integral extended to an orientable M-manifold for a covariant pseudotensor field $\overset{[W]}{A}_{i_1 i_2 \ldots i_{M-1}}$ of weight W can be furnished as

$$\int \overset{[W]}{A}_{i_1 i_2 \ldots i_M} e^{-W} d\tau^{i_1 i_2 \ldots i_M}. \tag{12}$$

Here we emphasize that the integral (12) is an absolute invariant.

In such a case Stokes' integral theorem (9) can be reformulated as follows

$$\int \partial_{i_M} \Big(\overset{[W]}{A}_{i_1 i_2 \ldots i_{M-1}} e^{-W} \Big) d\tau^{i_1 i_2 \ldots i_M} = \oint\limits_{\partial} \overset{[W]}{A}_{i_1 i_2 \ldots i_{M-1}} e^{-W} d\tau^{i_1 i_2 \ldots i_{M-1}}, \tag{13}$$

where the absolute tensor field is enclosed in parentheses.

It is obvious that the integrand in (13) can be transformed according to

$$\partial_{i_M} \Big(e^{-W} \overset{[W]}{A}_{i_1 i_2 \ldots i_{M-1}} \Big) = -W e^{-W} \frac{\partial_{i_M} e}{e} \overset{[W]}{A}_{i_1 i_2 \ldots i_{M-1}} + e^{-W} \partial_{i_M} \overset{[W]}{A}_{i_1 i_2 \ldots i_{M-1}}. \tag{14}$$

Substituting (14) into (13) we come to

$$-\int W e^{-W} \frac{\partial_{i_M} e}{e} \overset{[W]}{A}_{i_1 i_2 \ldots i_{M-1}} d\tau^{i_1 i_2 \ldots i_M} + \int e^{-W} \partial_{i_M} \overset{[W]}{A}_{i_1 i_2 \ldots i_{M-1}} d\tau^{i_1 i_2 \ldots i_M} =$$

$$= \oint\limits_{\partial} \overset{[W]}{A}_{i_1 i_2 \ldots i_{M-1}} e^{-W} d\tau^{i_1 i_2 \ldots i_{M-1}}. \tag{15}$$

On the other hand by virtue of

$$d\tau^{i_1 i_2 \ldots i_M} = d\tau^{[i_1 i_2 \ldots i_M]}$$

the integrand in (13) is transformed into

$$\partial_{i_M} \Big(\overset{[W]}{A}_{i_1 i_2 \ldots i_{M-1}} e^{-W} \Big) d\tau^{i_1 i_2 \ldots i_M} = \partial_{[i_M} \Big(\overset{[W]}{A}_{i_1 i_2 \ldots i_{M-1}]} e^{-W} \Big) d\tau^{i_1 i_2 \ldots i_M} =$$

$$= \nabla_{[i_M} \frac{\overset{[W]}{A}_{i_1 i_2 \ldots i_{M-1}]}}{e^W} d\tau^{i_1 i_2 \ldots i_M}. \tag{16}$$

Taking account of covariant constancy of integer powers of the fundamental orienting pseudoscalar, i.e.

$$\nabla_i e^m = \overset{[m]}{0} \quad (m = 0, \pm 1, \pm 2, \cdots),$$

equation (16) reads

$$\nabla_{[i_M} \frac{\overset{[W]}{A}_{i_1 i_2 \ldots i_{M-1}]}}{e^W} d\tau^{i_1 i_2 \ldots i_M} = e^{-W} \nabla_{[i_M} \overset{[W]}{A}_{i_1 i_2 \ldots i_{M-1}]} d\tau^{i_1 i_2 \ldots i_M}.$$

Finally, Stokes' integral theorem (13) is rewritten as

$$\int e^{-W} \nabla_{[i_M} \overset{[W]}{A}_{i_1 i_2 \ldots i_{M-1}]} d\tau^{i_1 i_2 \ldots i_M} d\tau^{i_1 i_2 \ldots i_M} = \oint\limits_{\partial} e^{-W} \overset{[W]}{A}_{i_1 i_2 \ldots i_{M-1}} d\tau^{i_1 i_2 \ldots i_{M-1}}, \tag{17}$$

which once more confirms the validity of the formulation (11).

By comparing the formulations of Stokes' theorem (15) and (17) we conclude that

$$\int e^{-W}\frac{\partial_{i_M} e}{e} \overset{[W]}{A}_{i_1 i_2 \ldots i_{M-1}} d\tau^{i_1 i_2 \ldots i_M} = 0. \tag{18}$$

It is then seen that equation (18) is satisfied by

$$\partial_i e = \overset{[+1]}{0} \quad (i = 1, 2, \ldots, N). \tag{19}$$

The introduced equation (19) can be interpreted, following the paper [10] by a choice of special coordinate systems constrained by the condition

$$e = \pm \overset{[+1]}{1}. \tag{20}$$

The equation (20) can be used for simplification of governing equations in mechanics of hemitropic micropolar continuum when the constitutive tensors are sensitive to space orientations. Note that if $e = -\overset{[+1]}{1}$ then there is an ambiguity in the calculation of the non-integer degrees e^{-W}.[2] The latter fact means that pseudotensors cannot have rational weights.

Note that the constraint $\sqrt{g} = \overset{[+1]}{1}$ following from (20) is often used not only in astronomy and the theory of relativity [11] but also in the mechanics of solids [12]. In the monograph [11, p. 135–142], the equation $\sqrt{g} = \overset{[+1]}{1}$ is used for the gravity equation derivation in 4-space-time, which greatly simplifies the field theory equations. In the monograph [12] the equation $\sqrt{g} = \overset{[+1]}{1}$ is systematically used while separating isostatic coordinates in the three-dimensional differential equations of the conventional mathematical theory of plasticity.

3. Different Stockes' integral theorem formulations for a two-dimensional manifold

Consider the orientations consistency procedure developed for Stokes' integral theorem formulation (9) in the case of a given 2-manifold. In the two-dimensional case, a 2-cell reper consists of the two enumerated vectors **1** and **2** as shown in Fig. 2. Let us transfer a continuously deforming reper from an interior point of the manifold to another point on the boundary contour, so that the reper directions remain linearly independent during the transfer. Let **s** be a vector directed outward relative to the 2-manifold and co-oriented to the vector $\bar{\mathbf{2}}$ (see Fig. 2). Thus, there are two possible cases. Vector $\bar{\mathbf{1}}$ defined by the 1-cell boundary reper can have two different directions corresponding to two ways of a 2-manifold boundary contour bypassing. In the first case (as can be seen from Fig. 2, *a*), the repers orientations $\bar{\mathbf{1}}$, $\bar{\mathbf{2}}$ and **1**, **2** are consistent as one is obtained from another by in-plane rotation. Since it can also be seen from Fig. 2, *a*, the boundary contour should be bypassed in the clockwise sense. In the second case, as it can be elucidated in Fig. 2, *b*, the repers $\bar{\mathbf{1}}$, $\bar{\mathbf{2}}$ and **1**, **2** have opposite orientations in virtue of one is obtained from another by mirroring the vector **1** into $\bar{\mathbf{1}}$. Apparently, the boundary contour is bypassed

[2]For example, if $e = -\overset{[+1]}{1}$ and $W = \dfrac{1}{2}$, then $e^W = \overset{[+1/2]}{i}$, which is impossible within the framework of real analysis.

in the counterclockwise sense. Note that in Stokes' integral theorem formulation (9), we should accept the orientations consistency as shown in Fig. 2, *a*, i.e. the boundary contour should be bypassed in the clockwise sense.

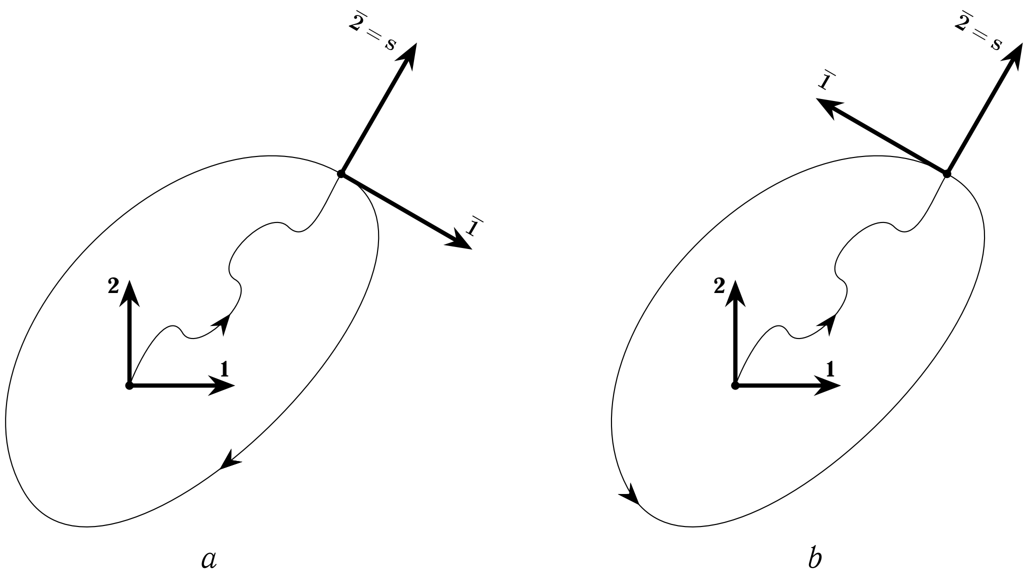

Fig. 2. Orientations consistency rules for tensor elementary volume in the case of 2-manifold according to the monograph [3]: *a* is the contour bypassing in the clockwise sense; *b* is the contour bypassing in the counterclockwise sense

The Stokes' integral theorem formulation (10) corresponds to orientations shown in Fig. 3. The vector field $\mathbf{s}$ is chosen co-oriented to the vector $\overline{\mathbf{1}}$, and the vector defined by a 1-cell boundary reper is denoted by $\overline{\mathbf{2}}$. Once again, there are two possible orientations of vector $\overline{\mathbf{2}}$. In the first case (as it can be seen from Fig. 3, *a*), the repers orientations $\overline{\mathbf{1}}$, $\overline{\mathbf{2}}$ and $\mathbf{1}$, $\mathbf{2}$ are opposite because one is obtained from another by mirroring the vector $\mathbf{2}$

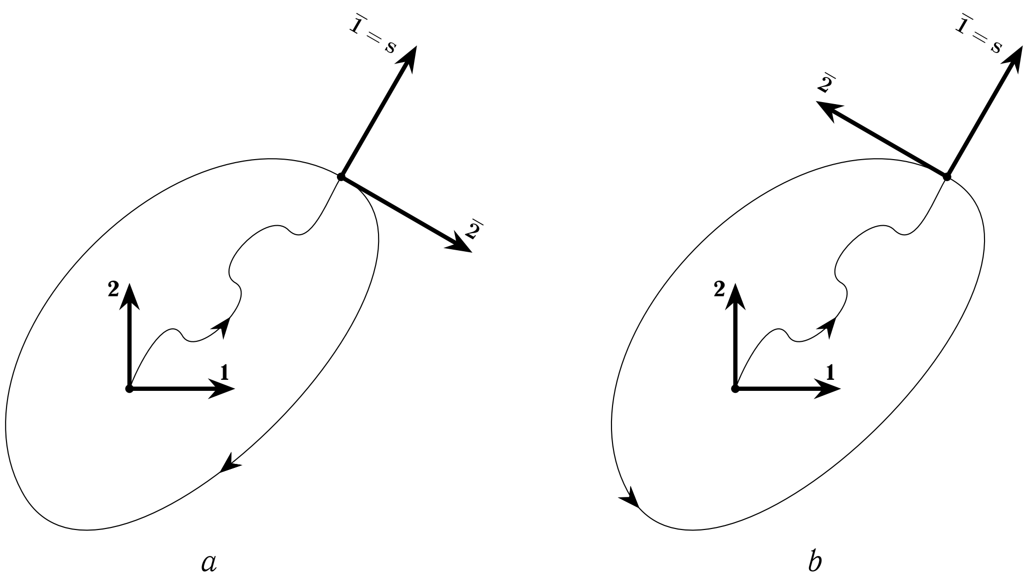

Fig. 3. Orientations consistency rules for tensor elementary volume in the case of 2-manifold according to the monographs [1, 2]: *a* is the contour bypassing in the clockwise sense; *b* is the contour bypassing in the counterclockwise sense

into $\overline{\mathbf{2}}$. In this case (as it can be also elucidated from Fig. 3, *a*) the boundary contour should be bypassed in the clockwise sense. In the second case, the reper $\overline{\mathbf{1}}$, $\overline{\mathbf{2}}$ can be obtained from the reper $\mathbf{1}$, $\mathbf{2}$ by in-plane rotation, so that they are co-oriented. As follows from Fig. 3, *b*, the boundary contour should be bypassed in the counterclockwise sense.

Note that, for example, in complex analysis, the positive direction of a loop is bypassed in the counterclockwise sense. Therefore, the formulation of Stokes' integral theorem in the form (10) is much more preferable in this case.

Conclusions

The present study is devoted to Stokes' integral theorem formulations for an asymmetric covariant pseudotensor field of a given weight and valency:

– the notions of orientable manifold and continuum playing an important role in micropolar elasticity have been discussed. In particular, this concerns the hemitropic micropolar elasticity;
– the fundamental tensors characterizing the metric and orientational properties of N-dimensional space have been introduced. The notion of the fundamental orienting pseudoscalar and its covariant constancy has been discussed;
– the rule providing a scheme of reducing pseudotensors to absolute ones has been given;
– the concepts of tensor and pseudotensor elementary volume (area) of M-cell have been revisited;
– the generalized pseudotensor formulations of Stokes' integral theorem have been refined for the class of special coordinate nets constrained by the equation $e^2 = \overset{[2]}{1}$;
– a procedure of elementary cells orientations consistency for Stokes' integral theorem formulations in the case of a 2-manifold has been considered and discussed. The preferable bypasses of boundary contours associated with boundary 1-cell direction for different formulations of Stokes' integral theorem have been discriminated and discussed.

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