Article

Forcing total outer connected monophonic number of a graph

K. Ganesamoorthy\textsuperscript{1}, S. Lakshmi Priya\textsuperscript{2}

\textsuperscript{1}Coimbatore Institute of Technology, Department of Mathematics, Coimbatore — 641 014, India
\textsuperscript{2}CIT Sandwich Polytechnic College, Department of Mathematics, Coimbatore — 641 014, India

Kathiresan Ganesamoorthy, kvgm_2005@yahoo.co.in, https://orcid.org/0000-0003-2769-1991
Shanmugam Lakshmi Priya, lakshmiuspriya@gmail.com, https://orcid.org/0000-0001-7367-1532

Abstract. For a connected graph $G = (V, E)$ of order at least two, a subset $T$ of a minimum total outer connected monophonic set $S$ of $G$ is a forcing total outer connected monophonic subset for $S$ if $S$ is the unique minimum total outer connected monophonic set containing $T$. A forcing total outer connected monophonic subset for $S$ of minimum cardinality is a minimum forcing total outer connected monophonic subset of $S$. The forcing total outer connected monophonic number $f_{tom}(S)$ in $G$ is the cardinality of a minimum forcing total outer connected monophonic subset of $S$. The forcing total outer connected monophonic number of $G$ is $f_{tom}(G) = \min\{f_{tom}(S)\}$, where the minimum is taken over all minimum total outer connected monophonic sets $S$ in $G$. We determine bounds for it and find the forcing total outer connected monophonic number of a certain class of graphs. It is shown that for every pair $a, b$ of positive integers with $0 \leq a < b$ and $b \geq a + 4$, there exists a connected graph $G$ such that $f_{tom}(G) = a$ and $cm_{to}(G) = b$, where $cm_{to}(G)$ is the total outer connected monophonic number of a graph.

Keywords: total outer connected monophonic set, total outer connected monophonic number, forcing total outer connected monophonic subset, forcing total outer connected monophonic number

© Ganesamoorthy K., Lakshmi Priya S., 2022
Acknowledgements: The first author’s research work was supported by National Board for Higher Mathematics (NBHM), Department of Atomic Energy (DAE), Government of India (project No. NBHM/R.P.29/2015/Fresh/157).


This is an open access article distributed under the terms of Creative Commons Attribution 4.0 International License (CC-BY 4.0)
Introduction

By a graph $G = (V, E)$ we mean a finite simple undirected connected graph. The order and size of $G$ are denoted by $p$ and $q$, respectively. For basic graph theoretic terminology we refer to Harary [1, 2]. The distance $d(x, y)$ between two vertices $x$ and $y$ in a connected graph $G$ is the length of a shortest $x – y$ path in $G$. An $x – y$ path of length $d(x, y)$ is called an $x – y$ geodesic. A vertex $v$ of a connected graph $G$ is called an endvertex of $G$ if its degree is 1. A vertex $v$ of a connected graph $G$ is called a support vertex of $G$ if it is adjacent to an endvertex of $G$. The neighborhood of a vertex $v$ is the set $N(v)$ consisting of all vertices $u$ which are adjacent with $v$. A vertex $v$ is an extreme vertex if the subgraph induced by its neighbors is complete. A chord of a path $P$ is an edge joining two non-adjacent vertices of $P$. A path $P$ is called a monophonic path if it is a chordless path. A set $S$ of vertices of $G$ is a monophonic set of $G$ if each vertex $v$ of $G$ lies on a $x – y$ monophonic path for some $x$ and $y$ in $S$. The minimum cardinality of a monophonic set of $G$ is the monophonic number of $G$ and is denoted by $m(G)$. The monophonic number of a graph, an algorithmic aspect of monophonic concepts was introduced and studied in [3–7]. A total monophonic set of a graph $G$ is a monophonic set $S$ such that the subgraph $G[S]$ induced by $S$ has no isolated vertices. The minimum cardinality of a total monophonic set of $G$ is the total monophonic number of $G$ and is denoted by $m_t(G)$. The total monophonic number of a graph and its related concepts were studied in [8–10]. A set $S$ of vertices in a graph $G$ is said to be an outer connected monophonic set if $S$ is a monophonic set of $G$ and either $S = V$ or the subgraph induced by $V – S$ is connected. The minimum cardinality of an outer connected monophonic set of $G$ is the outer connected monophonic number of $G$ and is denoted by $m_{oc}(G)$. The outer connected monophonic number of a graph was introduced in [11]. Very recently, outer connected monophonic concepts have been widely investigated in graph theory, such as a connected outer connected monophonic number [12], extreme outer connected monophonic graphs [13], and so on. A total outer connected monophonic set $S$ of $G$ is an outer connected monophonic set such that the subgraph induced by $S$ has no isolated vertices. The minimum cardinality of a total outer connected monophonic set of $G$ is the total outer connected monophonic number of $G$ and is denoted by $cm_{toc}(G)$.

The authors of this article introduced and studied the general externally total outer connected monophonic number of a graph and proved the following theorems\(^1\), which will be used further.

**Theorem 1.** Each extreme vertex and each support vertex of a connected graph $G$ belong to every total outer connected monophonic set of $G$.

**Theorem 2.** For the complete graph $K_p (p \geq 2)$, $cm_{toc}(K_p) = p$.

**Theorem 3.** For any non-trivial tree $T$, the set of all endvertices and support vertices of $T$ is the unique minimum total outer connected monophonic set of $G$.

\(^1\)Ganesamoorthy K., Lakshmi Priya S. The total outer connected monophonic number of a graph. *Transactions of A. Razmadze Mathematical Institute*, accepted.
Theorem 4. For any connected graph $G$, $cm_{to}(G) = 2$ if and only if $G = K_2$.

Throughout this paper, $G$ denotes a connected graph with at least two vertices.

1. Main Results

Definition 1. Let $S$ be a minimum total outer connected monophonic set of $G$. A subset $T$ of $S$ is a forcing total outer connected monophonic subset for $S$ if $S$ is the unique minimum total outer connected monophonic set containing $T$. A forcing total outer connected monophonic subset for $S$ of minimum cardinality is a minimum forcing total outer connected monophonic subset of $S$. The forcing total outer connected monophonic number $f_{tom}(S)$ in $G$ is the cardinality of a minimum forcing total outer connected monophonic subset of $S$. The forcing total outer connected monophonic number of $G$ is $f_{tom}(G) = \min\{f_{tom}(S)\}$, where the minimum is taken over all minimum total outer connected monophonic sets $S$ in $G$.

Example 1. For the graph $G$ in Fig. 1, it is clear that $S_1 = \{v_1, v_2, v_4, v_5\}$, $S_2 = \{v_1, v_4, v_5, v_8\}$, $S_3 = \{v_1, v_2, v_5, v_6\}$ and $S_4 = \{v_1, v_5, v_6, v_8\}$ are the minimum total outer connected monophonic sets of $G$. It is clear that no minimum total outer connected monophonic set $S_i (i = 1, 2, 3, 4)$ is the unique minimum total outer connected monophonic set containing any of its 1-element subsets. It is easy to see that $\{v_2, v_4\}$ is a forcing total outer connected monophonic subset contained in $S_1$ and $f_{tom}(S_1) = 2$. Hence, we have $f_{tom}(G) = 2$. By Theorem 3, for any non-trivial tree $T$, the set of all endvertices and support vertices of $T$ is the unique minimum total outer connected monophonic set of $T$ and so $f_{tom}(T) = 0$.

Theorem 5. For any connected graph $G$ of order $p$, $0 \leq f_{tom}(G) \leq cm_{to}(G) \leq p$.

Proof. By the definition of the forcing total outer connected monophonic number of a graph, it is clear that $f_{tom}(G) \geq 0$. Let $S$ be a minimum total outer connected monophonic set of $G$. Clearly, $f_{tom}(S) \leq |S| = cm_{to}(G)$ and $f_{tom}(G) = \min\{f_{tom}(S)\}$, where the minimum is taken over all minimum total outer connected monophonic sets $S$ in $G$. Hence $0 \leq f_{tom}(G) \leq cm_{to}(G) \leq p$. □

Remark 1. The bounds in Theorem 5 are sharp. By Theorem 3, for any non-trivial tree $T$, the set of all endvertices and support vertices of $T$ is the unique minimum total outer connected monophonic set of $T$ and so $f_{tom}(T) = 0$. By Theorem 2, for the complete graph $K_p (p \geq 2)$, $cm_{to}(K_p) = p$. Also all the inequalities in Theorem 5 can be strict. For the graph $G$ given in Fig. 1 of order 8, it is clear that no 2-element subset or 3-element subset of $V(G)$ is a total outer connected monophonic set of $G$. The minimum total outer connected monophonic sets of $G$ are $S_1 = \{v_1, v_2, v_4, v_5\}$, $S_2 = \{v_1, v_4, v_5, v_8\}$, $S_3 = \{v_1, v_2, v_5, v_6\}$ and $S_4 = \{v_1, v_5, v_6, v_8\}$ so that $cm_{to}(G) = 4$. It is clear that $f_{tom}(S_i) = 2 (i = 1, 2, 3, 4)$ and so $f_{tom}(G) = 2$. Thus $0 < f_{tom}(G) < cm_{to}(G) < p$.

The following theorem characterizes graphs $G$ for which the lower bound in Theorem 5 is attained and also characterizes graphs $G$ for which $f_{tom}(G) = 1$ and $f_{tom}(G) = cm_{to}(G)$.
Theorem 6. Let $G$ be a connected graph. Then

(i) \( f_{\text{tom}}(G) = 0 \) if and only if $G$ has the unique minimum total outer connected monophonic set;

(ii) \( f_{\text{tom}}(G) = 1 \) if and only if $G$ has at least two minimum total outer connected monophonic sets, one of which is the unique minimum total outer connected monophonic set containing one of its elements;

(iii) \( f_{\text{tom}}(G) = cm_{\text{to}}(G) \) if and only if no minimum total outer connected monophonic set of $G$ is the unique minimum total outer connected monophonic set containing any of its proper subsets.

Proof. (i) Let \( f_{\text{tom}}(G) = 0 \). Then, by the definition, \( f_{\text{tom}}(S) = 0 \) for some minimum total outer connected monophonic set $S$ of $G$ so that the empty set $\varnothing$ is the minimum forcing subset for $S$. Since the empty set $\varnothing$ is a subset of every set, it follows that $S$ is the unique minimum total outer connected monophonic set of $G$. The converse is clear.

(ii) Let \( f_{\text{tom}}(G) = 1 \). Then by (i), $G$ has at least two minimum total outer connected monophonic sets. Since \( f_{\text{tom}}(G) = 1 \), there is a 1-element subset $T$ of a minimum total outer connected monophonic set $S$ of $G$ such that $T$ is not a subset of any other minimum total outer connected monophonic set of $G$. Thus $S$ is the unique minimum total outer connected monophonic set containing one of its elements. The converse is clear.

(iii) Let \( f_{\text{tom}}(G) = cm_{\text{to}}(G) \). Then \( f_{\text{tom}}(S) = cm_{\text{to}}(G) \) for every minimum total outer connected monophonic set $S$ in $G$. Since any total outer connected monophonic set of $G$ needs at least two vertices, \( cm_{\text{to}}(G) \geq 2 \) and hence \( f_{\text{tom}}(G) \geq 2 \). Then by (i), $G$ has at least two minimum total outer connected monophonic sets, and so the empty set $\varnothing$ is not a forcing subset for any minimum total outer connected monophonic set of $G$. Since \( f_{\text{tom}}(G) = cm_{\text{to}}(G) \), no proper subset of $S$ is a forcing subset of $S$. Thus no minimum total outer connected monophonic set of $G$ is the unique minimum total outer connected monophonic set containing any of its proper subsets.

Conversely, the data implies that $G$ contains more than one minimum total outer connected monophonic set, and no subset of any minimum total outer connected monophonic set $S$ other than $S$, is a forcing subset for $S$. Hence it follows that \( f_{\text{tom}}(G) = cm_{\text{to}}(G) \). \[ \square \]

Definition 2. A vertex $v$ of $G$ is said to be a total outer connected monophonic vertex if $v$ belongs to every minimum total outer connected monophonic set of $G$.

Remark 2. If $G$ has the unique minimum total outer connected monophonic set $S$, then every vertex in $S$ is a total outer connected monophonic vertex of $G$. Also, if $x$ is an extreme vertex or a support vertex of $G$, then $x$ is a total outer connected monophonic vertex of $G$. For the graph $G$ given in Fig. 1, $v_1$ and $v_5$ are the total outer connected monophonic vertices of $G$.

The next theorem and corollary are an immediate consequence of the definitions of total outer connected monophonic vertex and a forcing total outer connected monophonic subset of $G$.

Theorem 7. Let $G$ be a connected graph and let $\Psi_{\text{tom}}$ be the set of relative complements of the minimum forcing total outer connected monophonic subsets in their respective minimum total outer connected monophonic sets in $G$. Then \( \cap_{F \in \Psi_{\text{tom}}} F \) is the set of all total outer connected monophonic vertices of $G$. 

Научный отдел
Corollary 1. Let $S$ be a minimum total outer connected monophonic set of $G$. Then no total outer connected monophonic vertex of $G$ belongs to any minimum forcing total outer connected monophonic subset of $S$.

Theorem 8. Let $M$ be the set of all total outer connected monophonic vertices of $G$. Then $f_{tot}(G) \leq cm_{tot}(G) - |M|$.

Proof. Let $S$ be any minimum total outer connected monophonic set of $G$. Then $cm_{tot}(G) = |S|$, $M \subseteq S$, and $S$ is the unique minimum total outer connected monophonic set containing $S - M$. Hence $f_{tot}(G) \leq |S - M| = |S| - |M| = cm_{tot}(G) - |M|$. ☐

Corollary 2. If $G$ is a connected graph with $l$ extreme vertices and $k$ support vertices, then $f_{tot}(G) \leq cm_{tot}(G) - (l + k)$.

Remark 3. The bound in Theorem 8 is sharp. For the graph $G$ given in Fig. 1, the minimum total outer connected monophonic sets of $G$ are $S_1 = \{v_1, v_2, v_4, v_5\}$, $S_2 = \{v_1, v_4, v_5, v_8\}$, $S_3 = \{v_1, v_2, v_5, v_6\}$, and $S_4 = \{v_1, v_5, v_6, v_8\}$ so that $cm_{tot}(G) = 4$. It is clear that $f_{tot}(S_i) = 2(i = 1, 2, 3, 4)$ and so $f_{tot}(G) = 2$. Also, $M = \{v_1, v_3\}$ is the set of all total outer connected monophonic vertices of $G$ and so $f_{tot}(G) = cm_{tot}(G) - |M|$. The inequality in Theorem 8 can be strict. For the graph $G$ given in Fig. 2, the minimum total outer connected monophonic sets of $G$ are $M_1 = \{v_1, v_2, v_3, v_6\}$, $M_2 = \{v_3, v_4, v_5, v_6\}$, $M_3 = \{v_2, v_3, v_4, v_6\}$, and $M_4 = \{v_2, v_3, v_4, v_6\}$ and so $cm_{tot}(G) = 4$. It is clear that $f_{tot}(M_i) = 1$ ($i = 1, 2$), and so $f_{tot}(G) = 1$. Also, the vertices $v_3$ and $v_6$ are the total outer connected monophonic vertices of $G$, we have $f_{tot}(G) < cm_{tot}(G) - |M|$.

Theorem 9. If $G$ is a connected graph with $cm_{tot}(G) = 2$, then $f_{tot}(G) = 0$.

Proof. If $cm_{tot}(G) = 2$ then by Theorem 4, we have $G = K_2$. Hence $V(G)$ is the unique minimum total outer connected monophonic set of $G$. Also, by Theorem 6(i), $f_{tot}(G) = 0$. ☐

Remark 4. The converse of Theorem 9 need not be true. For the path $P_4$ of order 4, the vertex set $V(P_4)$ is the unique minimum total outer connected monophonic set of $G$ and so $cm_{tot}(P_4) = 4$. By Theorem 6(i), $f_{tot}(P_4) = 0$.

Theorem 10. For the complete bipartite graph $G = K_{m,n}$,  

$$f_{tot}(G) = \begin{cases}  m + n - 1 & \text{if } 2m \leq n, \\  4 & \text{if } 3m \leq n. \end{cases}$$

Proof. Let $U = \{u_1, u_2, \ldots, u_m\}$ and $W = \{w_1, w_2, \ldots, w_n\}$ be the partite sets of $G$, where $m \leq n$. We prove this theorem by considering two cases.

Case 1. If $m = 2$, then it is clear that any minimum total outer connected monophonic sets of $G$ is of the form $V(G) - \{w_i\}$ ($1 \leq i \leq n$) or $V(G) - \{u_j\}$ ($1 \leq j \leq m$). It is easy to verify that, no minimum total outer connected monophonic set of $G$ is the unique
minimum total outer connected monophonic set containing any of its proper subsets. Then by Theorem 6 (iii), we have $f_{tom}(G) = m + n - 1$.

**Case 2.** If $3 \leq m \leq n$, then any minimum total outer connected monophonic set of $G$ is obtained by choosing any two elements from $U$ as well as $W$, and $G$ has at least two minimum total outer connected monophonic sets. Hence $cm_{to}(G) = 4$. Clearly, no minimum total outer connected monophonic set of $G$ is the unique minimum total outer connected monophonic set containing any of its proper subsets. Then by Theorem 6 (iii), we have $f_{tom}(G) = cm_{to}(G) = 4$.

**Theorem 11.** For any cycle $C_n(n \geq 3)$, $f_{tom}(C_n) = \begin{cases} 0 & \text{if } n = 3, \\ 3 & \text{if } n = 4, \\ 2 & \text{if } n \geq 5. \end{cases}$

**Proof.** Let $C_n : v_1, v_2, \ldots, v_n, v_1$ be a cycle of order $n$. We prove this theorem by considering two cases.

**Case 1:** $n = 3$. Since $C_3$ is the complete graph of order $3$, $V(C_3)$ is the unique minimum total outer connected monophonic set of $C_3$. By Theorem 6 (i), $f_{tom}(C_3) = 0$.

**Case 2:** $n \geq 4$. It is clear that no 2-element subset of $V(C_n)$ is a total outer connected monophonic set of $C_n$. It is easy to verify that any minimum total outer connected monophonic set of $C_n$ consists of three consecutive vertices of $C_n$ so that $cm_{to}(C_n) = 3$. For $n = 4$, it is clear that no minimum total outer connected monophonic set of $S$ of $C_n$ is the unique minimum total outer connected monophonic set containing any of its proper subsets. Thus by Theorem 6 (iii), we have $f_{tom}(C_4) = 3$. For $n \geq 5$, it is clear that the set of two non-adjacent vertices of any minimum total outer connected monophonic set $S$ of $C_n$ is a minimum forcing total outer connected monophonic subset of $S$ and so $f_{tom}(S) = 2$. Hence $f_{tom}(C_n) = 2$.

**Theorem 12.** For the wheel $W_n = K_1 + C_n-1$ ($n \geq 5$), $f_{tom}(W_n) = \begin{cases} 3 & \text{if } n = 5, \\ 2 & \text{if } n \geq 6. \end{cases}$

**Proof.** It is clear that no 2-element subset of $V(W_n)$ is a total outer connected monophonic set of $W_n$. It is easy to observe that any minimum total outer connected monophonic set of $W_n$ consists of three consecutive vertices of $C_n-1$ so that $cm_{to}(W_n) = 3$. For $n = 5$, it is clear that no minimum total outer connected monophonic set of $W_5$ is the unique minimum total outer connected monophonic set containing any of its proper subsets. Thus by Theorem 6 (iii), we have $f_{tom}(W_5) = 3$. For $n \geq 6$, it is clear that the set of two non-adjacent vertices of any minimum total outer connected monophonic set $S$ of $W_n$ is a minimum forcing total outer connected monophonic subset of $S$ and so $f_{tom}(S) = 2$. Hence $f_{tom}(W_n) = 2$.

**Theorem 13.** For any complete graph $G = K_p(p \geq 2)$ or any non-trivial tree $G = T$, $f_{tom}(G) = 0$.

**Proof.** Let $G = K_p$. By Theorem 2, the set of all vertices of $G$ is the unique minimum total outer connected monophonic set of $G$ and so by Theorem 6 (i), $f_{tom}(G) = 0$. If $G$ is a non-trivial tree, then by Theorem 3, the set of all endvertices and support vertices of $G$ is the unique minimum total outer connected monophonic set of $G$ and by Theorem 6 (i), $f_{tom}(G) = 0$.

**Theorem 14.** For every pair $a, b$ of integers such that $0 \leq a < b$ and $b \geq a + 4$, there is a connected graph $G$ with $f_{tom}(G) = a$ and $cm_{to}(G) = b$. 

284 Научный отдел
**Proof.** If \( a = 0 \), let \( G = K_a \). Then by Theorem 13, \( f_{\text{tom}}(G) = 0 \), and by Theorem 2, \( cm_{\text{to}}(G) = b \). Now, assume that \( 0 < a < b \). The required graph \( G \) is obtained from the star \( K_{1,a} \) having the vertex set \( \{z_1, z_2, z_3, z_4, z_5\} \) with \( z_5 \) as the cut-vertex by adding \( a + b - 2 \) new vertices \( w_1, w_2, \ldots, w_a, v_1, v_2, \ldots, v_a, u_1, u_2, \ldots, u_{b-a-3} \), and joining each \( w_i (1 \leq i \leq a) \) to the vertices \( z_2, z_1 \) and \( z_4 \); and joining each \( v_i (1 \leq i \leq a) \) to the vertices \( z_2, z_4 \) and \( z_5 \); and joining each \( u_i (1 \leq i \leq b-a-3) \) to the vertex \( z_5 \); and also joining the vertex \( x \) to the vertex \( z_1 \), the vertex \( z_1 \) to the vertex \( z_5 \), and the vertex \( z_2 \) to the vertex \( z_4 \). The graph \( G \) is shown in Fig. 3. Let \( S = \{u_1, u_2, \ldots, u_{b-a-3}, x, z_1, z_5\} \) be the set of all endvertices and support vertices of \( G \). By Theorem 1, every total outer connected monophonic set of \( G \) contains \( S \). It is clear that \( S \) is not a total outer connected monophonic set of \( G \). We observe that every minimum total outer connected monophonic set of \( G \) contains exactly one vertex from the set \( \{v_i, w_i\} \) for every \( (1 \leq i \leq a) \). Thus \( cm_{\text{to}}(G) \geq b \). Since \( S_1 = S \cup \{w_1, w_2, \ldots, w_a\} \) is a total outer connected monophonic set of \( G \), it follows that \( cm_{\text{to}}(G) = b \).

Next, we show that \( f_{\text{tom}}(G) = a \). Since every minimum total outer connected monophonic set of \( G \) contains \( S \), it follows from Theorem 8 that \( f_{\text{tom}}(G) \leq cm_{\text{to}}(G) - |S| = b - (b - a) = a \). It is clear that every minimum total outer connected monophonic set \( S' \) of \( G \) is of the form \( S' = \{x_1, x_2, \ldots, x_a\} \), where \( x_i \in \{v_i, w_i\} \) for every \( (1 \leq i \leq a) \). Let \( T \) be any proper subset of \( S' \) with \( |T| < a \). Then there is a vertex \( x \in S' - S \) such that \( x \notin T \). If \( x = v_i \) \((1 \leq i \leq a)\), then \( S'' = (S' - \{v_i\}) \cup \{w_i\} \) is a minimum total outer connected monophonic set of \( G \) containing \( T \). Similarly, if \( x = w_j (1 \leq j \leq a) \), then \( S'' = (S' - \{w_j\}) \cup \{v_j\} \) is a minimum total outer connected monophonic set of \( G \) containing \( T \). Thus \( S' \) is not the unique minimum total outer connected monophonic set containing \( T \) and so \( T \) is not a forcing total outer connected monophonic subset of \( S' \). This is true for all minimum total outer connected monophonic sets of \( G \) and so \( f_{\text{tom}}(G) = a \).

\[
\text{Fig. 3. A graph } G \text{ with } f_{\text{tom}}(G) = a > 0 \text{ and } cm_{\text{to}}(G) = b > a
\]

### References


Поступила в редакцию / Received 15.09.2021
Принята к публикации / Accepted 12.12.2021
Опубликована / Published 31.08.2022