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Article

Application of queueing network models in insurance

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Abstract. The purpose of this paper is to study the issues of the functioning of insurance companies using the methods of the queueing networks theory. The introduction provides a brief overview of scientific publications in this area. In particular, research based on the use of Markov stochastic processes and queueing systems are considered. In the first section of the article, a closed exponential queueing network is proposed as a model for the process of processing insurance claims. A detailed description of the corresponding network model is given. The stay of each job at a specific network node and its routing between the nodes correspond to the customer claim status in the insurance company and the process of its routing between claims adjusters of different types of risks. The process of changing the number of jobs at the nodes was studied under the asymptotic assumption of a large number of jobs in the second section of the article. In this case, its probability density function satisfies the Fokker – Planck – Kolmogorov equation. The system of differential equations for the first-order and second-order moments of the state vector was substantiated in the third section of the article. The solution of this system allows for predicting the dynamics of the expected number of insurance claims in the model nodes in both transient and steady states. Second-order moments can be used to calculate the variability of the number of insurance claims at the model nodes and to study the correlation between the number of claims at different nodes with time. The areas of implementation were considered.

Keywords: insurance company, mathematical model, queueing network, asymptotic analysis, insurance claims processing model
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Применение сетевых моделей массового обслуживания в страховании

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Аннотация. Целью работы является исследование вопросов функционирования страховых компаний с помощью методов теории сетей массового обслуживания. Во ведении дан краткий обзор научных работ, посвященных моделированию процессов функционирования страховых компаний. В частности, рассмотрены работы, основанные на применение марковских случайных процессов и систем массового обслуживания. В первой части статьи в качестве модели процесса обработки страховых исков предложена замкнутая экспоненциальная сеть массового обслуживания. Дано описание процесса и соответствующей ему сетевой модели. Пребыванию заявки в определенном узле сети массового обслуживания и ее маршрутизации между узлами сети соответствует определенное состояние страхового иска и его маршрутизация между оценщиками рисков разных типов в страховой компании. Во второй части статьи процесс состояния сети массового обслуживания исследован в асимптотическом случае большого числа заявок. В этом случае его плотность распределения вероятностей удовлетворяет уравнению Фоккера – Планка – Колмогорова. В третьей части статьи обоснован переход к системе обыкновенных дифференциальных уравнений для моментов первых двух порядков вектора состояния модели. Решение этой системы позволяет прогнозировать динамику ожидаемого числа страховых исков в узлах модели в стационарном и переходном режимах. Моменты второго порядка могут быть использованы для расчета волатильности числа страховых исков в каждом из узлов модели и для исследования корреляции между числом исков в разных узлах сети. В заключении отмечены области применения полученных результатов.

Ключевые слова: страховая компания, математическая модель, сеть массового обслуживания, асимптотический анализ, модель обработки страховый исков

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Introduction

At present, mathematical models in the field of the insurance industry and the operation of insurance companies are of great interest. The undeniable presence of risks and randomness entails the use of methods of probability theory and mathematical statistics for modeling insurance processes. In particular, the methods of the queueing networks theory can be used to study the issues of the functioning of insurance companies.

The classical dynamic theory of insurance is the most adequate for describing the operation of an insurance company. P. Lundberg is considered its founder [1], the presentation of this theory from the point of view of stochastic processes was carried out by H. Cramer in 1955. A description of various kinds of studies within the framework of the classical model can be found in monographs [2, 3]. From a theoretical point of view, the insurance process is presented as a model of a reservoir into which premiums flow in and insurance claims paid by the insurance company flow out. A distinguishing characteristic of this model is that the capital inflow is considered regular, and the capital expenditure is irregular. The main advantage of the classical model is its simplicity, which makes it possible to explicitly calculate various performance characteristics of an insurance company.

A large number of scientific works on the mathematical theory of insurance are devoted to the development and refinement of the classical model. The most famous lines of research are associated with finding and studying the risk of bankruptcy of an insurance company, premium calculating, and finding the optimal capital management strategy [4,5]. In the mathematical theory of insurance, Markov models and Markov processes are widely used [6]. We should note the works [7,8], where the mathematical model of an insurance company is presented in the form of a two-dimensional random process, the components of which are the company's capital and the number of insured risks. In [7,8], models with a limited number of potential customers and models in which the number of new customers linearly depends on the number of those already insured are considered.

The flows of risks arriving at the insurance company at random times and random time intervals required for their processing predetermined the need to use queueing theory methods to develop mathematical models of claims processing processes. The study of risk processes by methods of queueing theory is elegantly presented in [9]. An infinite-server queueing system is proposed as a mathematical model of an insurance company in [10]. The incoming flow of customers is considered Poisson. Service by each of the servers is recurrent and equally distributed, the service time simulates the duration of the insurance contract and, thus, the time the client is in the company. Assuming the size of insurance premiums and the size of payments to be independent random variables with certain expected values, the average capital of the company is analyzed. Queueing networks were used as models for the processes of handling claims and premiums in an insurance company [11,12].

In this paper, a closed exponential queueing network is considered as a model of insurance claims processing.

1. Model description

Our focus in this article is on a large insurance company with a limited number of clients (insured, policyholders), denote this number by K. A closed exponential queueing network with nodes S_i, $i = \overline{0, n+1}$, is proposed as a model of insurance claims processing. The node S_i is a $\cdot/M/m_i$ type queueing system, $i = \overline{1, n+1}$. K-linear hypothetical system S_0 plays the role of an external environment and is considered as a dependable source of arriving claims. Each insurance claim corresponds to the job arrival in the queueing network. The stay of the job at a specific network node and its routing between the nodes correspond to the status of a claim in the insurance company and the process of its routing between claims adjusters of different kinds of risks.

Consider an insurance company that insures policyholders against n kinds of risks. Each insured can be in one of the following states: S_0 — no need to file a claim to the insurer, S_{n+1} — a claim is required payment by one of m_{n+1} cashiers, S_i — a claim is required service by one of the m_i claims adjusters processing risks of the ith kind, $i = \overline{1, n}$, for example, auto, life, health, credit, property and other types of risks. Accordingly, each job of the queueing network can be located in one of the queueing systems S_i, $i = \overline{0, n+1}$.

The claim transition from state S_0 to state S_i corresponds to the claim submitted by the insured to the insurer for processing by the type i claim adjusters, which occurs at a random time, regardless of the state of other policyholders, but depends on the time. The probability of this transition in the time interval $[t, t+\Delta t]$ is equal to $\lambda(t) p_{0i} \Delta t + o(\Delta t)$, where $\lambda(t)$ is the arrival rate of claims. After the end of claim processing in state

S_i, $i = \overline{1, n}$, the insured goes to S_{n+1}, where the claim is paid. From state S_{n+1} the claim is routed to state S_0. The served claims transition matrix is a stochastic matrix $(p_{ij})_{(n+2)\times(n+2)}$ with the following nonzero entries: p_{0i}, $\sum\limits_{i=1}^{n} p_{0i} = 1$; $p_{i(n+1)} = 1$, $i = \overline{1, n}$; $p_{(n+1)0} = 1$. If at least one of the claims adjusters (servers) of system S_i is idle, then the incoming claim (job) immediately begins to be served at the moment of transition to state S_i, $i = \overline{1, n+1}$. Claims are handled on a first come first served basis. For the queueing network, this means that service requests for service are selected accordingly to the discipline FIFO. All m_i agents (servers) at the state (node) S_i are identical and they have exponentially distributed service time with rate μ_i, i.e., μ_i^{-1} is the mean service time, $i = \overline{1, n+1}$.

Let $k_i(t)$ denote the number of claims in the state S_i at the time t, $i = \overline{1, n+1}$. The allocation of claims according to possible states at time t fully describes the situation in the insurance company at that time. Accordingly, the allocation of jobs by queueing nodes completely determines the state of the queueing network model. Thus, the state of the network model under study at time t is represented by a vector

$$k(t) = (k_1(t), k_2(t), ..., k_{n+1}(t)) \, .$$

Taking into account the above-described, the process of changing the number of jobs at the nodes $k(t)$ is a continuous-time Markov chain with a unit state space. It is clear that the number of customers in the external node S_0 is $k_0(t) = K - \sum\limits_{i=1}^{n+1} k_i(t)$.

The purpose of the study is to derive the system of differential equations for the first-order and second-order moments of the vector $k(t)/K$ in the asymptotic case of large K.

2. Asymptotic analysis of the network model

Asymptotic analysis implies an approximation method of queueing network study under the assumption of a large number of jobs [13]. As mentioned above, the state of model $k(t)$ is a continuous-time discontinuous-state Markov process. Consider the passage to the limit from a Markov chain $k(t)$ to a continuous-state Markov process $\xi(t) = k(t)/K$ when K tends to be very large [14]. In contrast to discontinuous processes, continuous processes in any small time interval $\Delta t \to 0$ have some small change in the state $\Delta x \to 0$.

Denote I_i as n-vector with zero components excluding i-th, that is equal to 1. Having regard to all possible changes of $k(t)$ in the short time $\Delta t \to 0$, using the law of total probability, we pass to the Kolmogorov difference-differential equation for the probability $P(k, t) = P(k(t) = k)$:

$$\frac{dP(k,t)}{dt} = \sum_{i=1}^{n} \mu_i \min(m_i, k_i(t)) \left[P(k + I_i - I_{n+1}, t) - P(k, t)\right] +$$
$$+ \sum_{i=1}^{n} \mu_i \left[\min(m_i, k_i(t) + 1) - \min(m_i, k_i(t))\right] P(k + I_i - I_{n+1}, t) +$$
$$+ \mu_{n+1} \min(m_{n+1}, k_{n+1}(t)) \left[P(k + I_{n+1}, t) - P(k, t)\right] +$$
$$+ \mu_{n+1} \left[\min(m_{n+1}, k_{n+1}(t) + 1) - \min(m_{n+1}, k_{n+1}(t))\right] P(k + I_{n+1}, t) +$$

$$+\sum_{i=1}^{n}\lambda(t)p_{0i}\left(K-\sum_{j=1}^{n+1}k_j(t)\right)[P(k-I_i,t)-P(k,t)]+\sum_{i=1}^{n}\lambda(t)p_{0i}P(k-I_i,t).$$

In the general case, it is difficult to solve this system analytically. Insurance companies have a large number of insured. In connection with this, consider the important asymptotic case of a large number of insured $K>>1$ and find the probability distribution of the state vector $\xi(t)=k(t)/K$ of the insurance company model with an accuracy of $O(\varepsilon^2)$, where $\varepsilon=1/K$. Let $l_i=m_i/K$. Using the technique of asymptotic analysis of queueing networks described in articles [11–13, 15], we can prove the following theorem.

Theorem 1. *In the asymptotic case of a large number of jobs K the probability density function $p(x,t)$ of the random process $\xi(t)=\left(\frac{k(t)}{K}\right)=\left(\frac{k_1(t)}{K},\frac{k_2(t)}{K},...,\frac{k_{n+1}(t)}{K}\right)$ provided that it is differentiable with respect to t and twice continuously differentiable with respect to x_i, $i=\overline{1,n+1}$, satisfies up to $O(\varepsilon^2)$, where $\varepsilon=\frac{1}{K}$, the multidimensional Fokker – Planck – Kolmogorov equation*

$$\frac{\partial p(x,\ t)}{\partial t}=-\sum_{i=1}^{n+1}\frac{\partial}{\partial x_i}\left(A_i(x,t)p(x,t)\right)+\frac{\varepsilon}{2}\sum_{i,j=1}^{n+1}\frac{\partial^2}{\partial x_i\partial x_j}\left(B_{ij}(x,t)p(x,t)\right),$$

with drifts

$$A_i(x,t)=\sum_{j=1}^{n+1}\mu_j(p_{ji}-\delta_{ji})\min(l_j,\ x_j)+\lambda(t)p_{0i}\left(1-\sum_{j=1}^{n+1}x_j\right)$$

and diffusion coefficients (δ_{ij} is the Kronecker delta)

$$B_{ij}(x,t)=-\mu_ip_{ij}\min(l_j,x_j),\quad i\neq j;$$
$$B_{ii}(x,t)=\sum_{j=1}^{n+1}\mu_j(p_{ji}+\delta_{ji})\min(l_j,\ x_j)+\lambda(t)p_{0i}\left(1-\sum_{j=1}^{n+1}x_j\right),\quad i,j=\overline{1,n+1}.$$

3. The system of differential equations for the first-order and second-order moments of the state vector

It was proved in [16] that the first-order and second-order moments of the state vector elements $\xi_i(t)$ are determined with an accuracy of $O(1/K^2)$ from a system of ordinary differential equations

$$\frac{d\nu_i^{(1)}(t)}{dt}=\frac{dE\left(\xi_i(t)\right)}{dt}=E\left(A_i(\xi(t),t)\right),\quad i=\overline{1,n+1};\tag{1}$$
$$\frac{d\nu_{ij}^{(1,1)}(t)}{dt}=\frac{dE\left(\xi_i(t)\xi_j(t)\right)}{dt}=E\left(\xi_i(t)A_j(\xi(t),t)\right)+$$
$$+E\left(\xi_j(t)A_i(\xi(t),t)\right)+\varepsilon E\left(B_{ij}(\xi(t),t)\right),\quad i=\overline{1,n+1},\quad j=\overline{1,n+1};$$

$A_i(x,t)$ are drifts, $B_{ij}(x,t)$ are diffusion coefficients given in Theorem 1, $i,j=\overline{1,n+1}$.

For example, taking into account the transition matrix of the network model, the system (1) for the first-order moments of the state vector components is as follows

$$\begin{gathered}\frac{d\nu_i^{(1)}(t)}{dt} = -\mu_i \min(l_i, \nu_i^{(1)}(t)) + \lambda(t) p_{0i} \left(1 - \sum_{j=1}^{n+1} \nu_j^{(1)}(t)\right), i = \overline{1, n}, \\ \frac{d\nu_{n+1}^{(1)}(t)}{dt} = -\mu_{n+1} \min(l_{n+1}, \nu_{n+1}^{(1)}(t)) + \sum_{i=1}^{n} \mu_i \min(l_i, \nu_i^{(1)}(t)).\end{gathered} \tag{2}$$

The solution of the system (2) with a certain initial condition makes it possible to predict the expected number of claims in each model state. To solve (2), it is possible to use numerical methods or to solve the system analytically in the linearity regions of the right-hand side.

It is well established that under certain conditions, the network reaches a “steady state”. The steady-state means $\nu_i^{(1)} = \lim\limits_{t\to\infty} \nu_i^{(1)}(t)$, where $i = \overline{1, n+1}$, can be found from a system of algebraic balance equations of the form (2), where the left-hand sides are equal to zero. In this case, the arrival rate should be constant: $\lambda(t) = \lambda$. The relative number $\nu_i^{(1)}$ is numerically equal to the probability of the claim staying at the state (node) S_i, $i = \overline{1, n+1}$, in the steady state. Second-order moments can be used to calculate the variability of the number of customers at the nodes and to study the correlation between the number of customers at different nodes with time.

Conclusion

In this paper, the network stochastic model for the process of insurance claims handling was presented as a queueing network. The model was investigated in the asymptotic case of a large number of insured. This allows us to calculate the expected number and variability of claims at each processing stage. The method is applicable both in transient and steady-state. Other performance indicators can be found using mathematical methods for calculating the nodal characteristics of queueing networks. The areas of implementation of research results are the design of the process of insurance claims handling, improving its performance, and regulating its parameters.

References

1. Cramer H. Historical review of Filip Lundberg's works on risk theory. *Scandinavian Actuarial Journal*, 1969, vol. 52, pp. 6–12. https://doi.org/10.1080/03461238.1969.10404602
2. Gerber H. U. *Life Insurance Mathematics*. Berlin, Springer-Verlag, 1995. 220 p. https://doi.org/10.1007/978-3-662-03153-7
3. Grandell J. *Aspects of Risk Theory*. New York, Springer-Verlang, 1991. 175 p. http://dx.doi.org/10.1007/978-1-4613-9058-9
4. Bening V. E., Korolev V. Yu. Nonparametric estimation of the ruin probability for generalized risk processes. *Theory of Probability and its Applications*, 2003, vol. 47, iss. 1, pp. 1–16. https://doi.org/10.4213/tvp2954
5. Rotar V. I., Bening V. E. An introduction to the mathematical theory of insurance. *Surveys on Applied and in Industrial Mathematics*, 1994, vol. 1, iss. 5, pp. 698–779 (in Russian).
6. Abdyusheva S. R., Spivak S. I. Markov models in actuarial analysis. *Middle Volga Mathematical Society Journal*, 2003, vol. 5, iss. 1, pp. 224–232 (in Russian). EDN: XWOCZN
7. Glukhova E. V., Zmeyev O. A., Livshits K. I. *Mathematical Models of Insurance*. Tomsk, Tomsk State University Publ., 2004. 180 p. (in Russian).

8. Akhmedova D. D., Zmeyev O. A., Terpugov A. F. Optimization of activity of insurance company with account of expenses for advertising. *Tomsk State University Journal*, 2002, iss. 275, pp. 181–184 (in Russian). EDN: OYCDID
9. Asmussen S. *Applied Probability and Queues*. New York, Springer-Verlag, 2003. 438 p. https://doi.org/10.1007/b97236
10. Katz V. M., Livshits K. I., Nazarov A. A. Investigation of nonstationary interminable linear queueing system and their application for mathematical model of insurance company. *Tomsk State University Journal*, 2002, iss. 275, pp. 189–192 (in Russian). EDN: OYCDIX
11. Matalytski M., Rusilko T., Pankov A. Asymptotic analysis of the closed queueing structure with time-dependent service parameters and single-type messages. *Journal of Applied Mathematics and Computational Mechanics*, 2013, vol. 12, iss. 2, pp. 73–80. https://doi.org/10.17512/jamcm.2013.2.09
12. Matalytskiy M. A., Romanyuk T. V. Mathematical analysis of stochastic models of processing claims of various types in insurance companies. *Doklady of the National Academy of Sciences of Belarus*, 2005, vol. 49, iss. 1, pp. 18–23 (in Russian).
13. Medvedev G. A. About optimization of closed queueing systems. *Proceedings of the USSR Academy of Sciences. Engineering Cybernetis*, 1975, no. 6, pp. 65–73 (in Russian). EDN: YHFVDR
14. Tikhonov V. I., Mironov M. A. *Markov Processes*. Moscow, Sovetskoe Radio, 1977. 488 p. (in Russian).
15. Rusilko T. V., Matalytski M. A. *Queuing Network Models of Claims Processing in Insurance Companies*. Saarbrucken, LAP Lambert Academic Publishing, 2012. 336 p. (in Russian).
16. Rusilko T. V. The first two orders moments determination method for the state vector of the queueing network in the asymptotic case. *Vesnik of Yanka Kupala State University of Grodno. Series 2*, 2021, vol. 11, iss. 2, pp. 152–161 (in Russian). EDN: EZQQOM

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