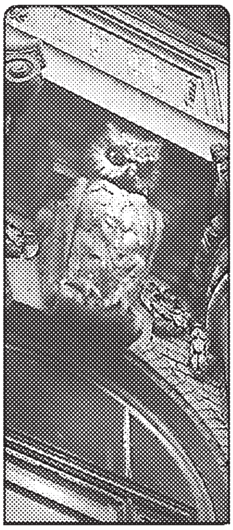

Научный отдел

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Article

## Risky investments and survival probability in the insurance model with two-sided jumps: Problems for integrodifferential equations and ordinary differential equation and their equivalence

**T. A. Belkina**[1✉], **A. S. Ogareva**[2]

[1]Central Economics and Mathematics Institute RAS, 47 Nakhimovsky prosp., Moscow 117418, Russia
[2]Moscow School of Economics of Lomonosov Moscow State University (MSE MSU), 1, str. 61 Leninskie Gory, Moscow 119234, Russia

**Tatiana A. Belkina**, tati.belkina@gmail.com, https://orcid.org/0000-0001-7384-0025, AuthorID: 12222
**Anna S. Ogareva**, ogareva-a@mail.ru

**Abstract.** We consider a model of an insurance portfolio that includes both non-life and life annuity insurance while assuming that the surplus (or some of its fraction) is invested in risky assets with the price dynamics given by a geometric Brownian motion. The portfolio surplus (in the absence of investments) is described by a stochastic process involving two-sided jumps and a continuous drift. Downward jumps correspond to the claim sizes and upward jumps are interpreted as random gains that arise at the final moments of the life annuity contracts realizations (i.e. at the moments of the death of policyholders). The drift is determined by the difference between premiums in the non-life insurance contracts and the annuity payments. We study the ruin problem for the model with investment using an approach based on integrodifferential equations (IDE) for the survival probabilities as a function of initial surplus. The main problem in calculating the survival probability as a solution of the IDE is that the initial value of the probability itself or its derivative at a zero initial surplus is priori unknown. For the case of the exponential distributions of the jumps, we propose a solution to this problem based on the assertion that the problem for an IDE is equivalent to a problem for an ordinary differential equation (ODE) with some nonlocal condition added. As a result, a solution to the original problem can be obtained as a solution to the ODE problem with an unknown parameter, which is finally determined using the specified nonlocal condition and a normalization condition.

**Keywords:** insurance, two-sided jumps, investments, risky asset, ruin problem, survival probability, integrodifferential equation

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# Рисковые инвестиции и вероятность неразорения в модели страхования с двусторонними скачками: задачи для интегродифференциальных уравнений и обыкновенного дифференциального уравнения и их эквивалентность

**Т. А. Белкина**[1]✉, **А. С. Огарева**[2]

[1]Центральный экономико-математический институт РАН, Россия, 117418, г. Москва, Нахимовский просп., д. 47

[2]Московская школа экономики МГУ имени М. В. Ломоносова, Россия, 119234, г. Москва, Ленинские горы, д. 1, стр. 61

**Белкина Татьяна Андреевна**, кандидат физико-математических наук, ведущий научный сотрудник, tati.belkina@gmail.com, https://orcid.org/0000-0001-7384-0025, AuthorID: 12222

**Огарева Анна Сергеевна**, студент магистратуры, ogareva-a@mail.ru

**Аннотация.** Рассматривается модель страхового портфеля, включающего рисковое страхование и пожизненные аннуитеты в предположении, что резерв (или некоторая его доля) инвестируется в рисковый актив, динамика цены которого моделируется геометрическим броуновским движением. Резерв портфеля (в отсутствие инвестиций) описывается стохастическим процессом, включающим двусторонние скачки и непрерывный снос, при этом скачки вниз соответствуют размерам требований, а скачки вверх интерпретируются как случайные доходы, возникающие в финальные моменты реализации аннуитетов (т.е. в моменты окончания жизни страхователей). Снос определяется разностью между премиями по рисковому страхованию и выплатами по аннуитетам. Проблема разорения в модели с инвестициями изучается с помощью подхода, основанного на интегродифференциальных уравнениях (ИДУ) для вероятности неразорения как функции начального резерва. Основная трудность при вычислении вероятности неразорения как решения ИДУ состоит в том, что начальные значения самой вероятности или ее производной (т.е. при нулевом начальном резерве) априорно в общем случае неизвестны. Для случая экспоненциального распределения скачков предлагается решение данной проблемы, основанное на утверждении об эквивалентности задачи для ИДУ задаче для обыкновенного дифференциального уравнения (ОДУ) при добавлении некоторого нелокального условия. В результате применения такого подхода может быть получено решение исходной задачи как решение задачи для ОДУ с неизвестным параметром, который в конечном итоге определяется при использовании указанного нелокального условия и условия нормировки.

**Ключевые слова:** страхование, двусторонние скачки, инвестиции, рисковый актив, проблема разорения, вероятность неразорения, интегродифференциальное уравнение

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## Introduction

A compound Poisson risk model involving two-sided jumps without investment, is considered in [1, 2]. In the context of ruin theory, the model can be interpreted as the surplus process of a business enterprise or an insurance company that is subject to random gains and losses. In [1], a Brownian perturbation is also added to this process, and the asymptotic estimate for the probability of ruin is obtained under some assumptions on the density functions of jumps. In [2] so-called Gerber – Shiu expected discounted penalty function in the model with a continuous downward drift is studied. The ruin problem for an insurance company having two business branches, life insurance and non-life insurance, and investing its reserves into a risky asset with the price dynamics given by a geometric Brownian motion, is investigated in [3]. In contrast to the models considered in [1] and [2], in [3] the processes of receipt of claims and random gains are modeled by two independent compound Poisson processes with different intensities. One of these processes involves random claims in non-life insurance (negative jumps), and the other process involves random gains in life insurance (positive jumps).

For the case of exponentially distributed jumps, it was shown in [3] that the survival probability is a solution of an ordinary differential equation of the fourth order; moreover, a power asymptotic representation for the survival probability as a solution of this equation was obtained when the initial surplus tends to infinity. Note that the mentioned asymptotic representation has an unknown multiplier and only gives an understanding of how fast the survival probability tends to one. The main goal of the present paper is to prove that the survival probability is a solution to a well-posed ODE problem with boundary conditions and some nonlocal conditions. This makes possible not only a qualitative but also a quantitative study of the survival probability in the future.

## 1. The model description and statement of the problem

We will consider an insurance portfolio that combines surpluses for two types of insurance businesses: life and non-life insurance. We propose that the typical life insurance contract for the policyholder is the life annuity with the subsequent transfer of its property to the benefit of the insurance company. Non-life insurance contracts have the same structure as in the classical Cramer – Lundberg model. Then the total portfolio surplus is of the form

$$R_t = u + \sum_{i=1}^{N_1(t)} C_i - \sum_{j=1}^{N(t)} Z_j + ct, \quad t \geqslant 0. \tag{1}$$

Here $R_t$ is the total portfolio surplus at time $t \geqslant 0$; $u$ is the initial surplus, $c \neq 0$ is the difference between the premium rate in non-life insurance and the life annuity rate (or the pension payments per unit of time), assumed to be deterministic and fixed. $N_1(t)$ is a homogeneous Poisson process with intensity $\lambda_1 > 0$ that, for any $t > 0$, determines the number of random revenues up to the time $t$; $C_k$ $(k = 1, 2, ...)$ are independent identically distributed (i.i.d.) random variables (r.v.) with a distribution function $G(z)$ $(G(0) = 0,\ \mathbf{E}C_1 = n < \infty,\ m > 0)$ that determine the revenue sizes (premiums) and are assumed to be independent of $N_1(t)$. These random revenues arise at the final moments of the life annuity contracts realizations. Further, $N(t)$ is a homogeneous Poisson process with intensity $\lambda > 0$ that, for any $t > 0$, determines the number of claims up to the time $t$; $Z_k$ $(k = 1, 2, ...)$ are i.i.d. r.v. with a distribution function $F(z)$ $(F(0) = 0,$ $\mathbf{E}Z_1 = m < \infty,\ m > 0)$ that determine the claim sizes and are assumed to be independent of $N(t)$. In addition, we assume that the processes of total premiums and total payments are independent.

We also assume that a fraction $\alpha$ of the surplus is invested at the time $t$ into a risky asset whose price follows a geometric Brownian motion $dS_t = \mu S_t dt + \sigma S_t dB_t$, where $\mu$ is the stock return rate, $\sigma > 0$ is the volatility, and $B_t$ is a standard Brownian motion independent of $N(t)$, $N_1(t)$, $C_i$'s and $Z_i$'s. The fraction $(1 - \alpha)$ of the surplus is invested in the risk-free asset whose price is governed by $dP_t = rP_t dt$, where $r$ is the risk-free interest rate; we suppose that $0 \leqslant r < \mu$.

Then the resulting surplus process $X_t$ is governed by the equation

$$dX_t = [(\alpha\mu + (1 - \alpha)r)\, dt + \alpha\sigma\, dw_t]X_t + dR_t, \qquad t \geqslant 0, \tag{2}$$

with the initial condition $X_0 = u$, where $R_t$ is defined by (1).

Note that if $0 < \alpha \leqslant 1$, then the insurance company purchases the risky asset at a cost of no more than its current surplus; if $\alpha > 1$, the insurance company borrows to invest in the risky asset; and if $\alpha < 0$, the insurance company shortsells the risky asset to invest in the risk-free asset.

Denote by $\varphi(u)$ the survival probability: $\varphi(u) = \mathbf{P}(X_t \geqslant 0,\ t \geqslant 0)$. Let us change the parameters of the assets as follows

$$a = \alpha\mu + (1 - \alpha)r, \quad b = \alpha\sigma, \tag{3}$$

and introduce the following assumptions: **(A1)** $b \neq 0$; **(A2)** $\rho := 2a/b^2 > 1$.

If assumptions **(A1)** and **(A2)** are satisfied, then a corresponding asset portfolio includes risky assets and the expected return on this portfolio is positive; moreover, assumption **(A2)** excludes unreliable asset portfolios leading to bankruptcy with a probability one (see [3]). Note that if assumptions **(A1)** and **(A2)** hold, then $\alpha$ satisfies the conditions $\alpha \neq 0$, $\alpha^- < \alpha < \alpha^+$, where $\alpha^- = [R - \sqrt{R^2 + 2r}]/\sigma \leqslant 0$, $\alpha^+ = [R + \sqrt{R^2 + 2r}]/\sigma > 0$, and $R$ is a Sharp ratio, i.e., $R = (\mu - r)/\sigma$.

If the survival probability $\varphi(u)$ for the process (2) belongs to the space $\mathcal{C}^2(\mathbb{R}_+)$ of twice continuously differentiable on $(0, \infty)$ functions, then the using of the Itô's formula and the total probability formula leads to the IDE

$$(b^2/2)u^2\varphi''(u) + (au + c)\varphi'(u) - \lambda \left[\varphi(u) - \int_0^u \varphi(u-x)dF(x)\right] - \\ -\lambda_1 \left[\varphi(u) - \int_0^\infty \varphi(u+y)dG(y)\right] = 0, \quad u > 0, \qquad (4)$$

where $a, b$ are defined in (3).

## 2. IDE and ODE problems in the case of the exponential distributions of jumps: formulations and equivalence

In the following we will assume that all the jumps are exponentially distributed, i.e., $F(x) = 1 - \exp(-x/m)$, $m > 0$, $G(y) = 1 - \exp(-y/n)$, $n > 0$. In this case, for $f \in \mathcal{C}^2(\mathbb{R}_+)$, IDE (4) can be rewritten in the following form:

$$(b^2/2)u^2 f''(u) + (au+c)f'(u) - \lambda[f(u) - (J_m f)(u)] - \lambda_1[f(u) - (J_{1,n} f)(u)] = 0, \quad u > 0, \qquad (5)$$

where the operators $J_m$, $J_{1,n} : \mathcal{C}[0,\infty) \to \mathcal{C}[0,\infty)$, are defined as follows:

$$(J_m f)(u) = \frac{1}{m}\int_0^u f(u-x)\exp(-x/m)\,dx = \frac{1}{m}\int_0^u f(s)\exp(-(u-s)/m)\,ds, \qquad (6)$$

$$(J_{1,n} f)(u) = \frac{1}{n}\int_0^\infty f(u+y)\exp(-y/n)\,dy = \frac{1}{n}\int_u^\infty f(s)\exp(-(s-u)/n)\,ds. \qquad (7)$$

Note that

$$(J_m f)'(u) = [f(u) - (J_m f)(u)]/m, \quad (J_{1,n} f)'(u) = [(J_{1,n} f)(u) - f(u)]/n. \qquad (8)$$

Denote by $g$ the left-hand side of the equation (5). Formally differentiating this function two times, we compose a linear combination of $g$ and its derivatives in such a way as to eliminate the integral components. For this purpose, we use the relations (8). If there is a solution to an equation (5) that belongs to $\mathcal{C}^4(\mathbb{R}_+)$, then it also satisfies the equation $g(u) + (m-n)g'(u) - nmg''(u) = 0$. The last equation is an ODE of the following form (see also [3]):

$$\frac{b^2}{2}u^2 f''''(u) + \left[c + (2b^2 + a)u + \frac{b^2(n-m)}{2mn}u^2\right] f'''(u) + \\ + \left[b^2 + 2a - \lambda - \lambda_1 + \frac{c(n-m)}{mn} + \frac{(b^2+a)(n-m)}{mn}u - \frac{b^2}{2mn}u^2\right] f''(u) + \\ + \left[\frac{a(n-m) + \lambda m - \lambda_1 n - c}{mn} - \frac{a}{mn}u\right] f'(u) = 0, \quad u > 0. \qquad (9)$$

Our main goal here is to establish conditions under which the solution of ODE (9) is also a solution of IDE(4) and determines the survival probability in the original problem.

Let us consider (formally at first) IDE (5) along with the conditions

$$\lim_{u\to+0} |f(u)| < \infty, \qquad (10)$$

$$\mathrm{I}\{c < 0\} \lim_{u\to+0} f(u) = 0, \qquad (11)$$

$$\lim_{u\to+0} |f'(u)| < \infty, \quad \lim_{u\to+0} [uf''(u)] = 0, \tag{12}$$

$$\lim_{u\to\infty} f(u) = 1, \tag{13}$$

$$\lim_{u\to\infty} [uf'(u)] = 0, \quad \lim_{u\to\infty} [u^2 f''(u)] = 0, \tag{14}$$

$$0 \leqslant f(u) \leqslant 1, \quad u \in \mathbb{R}_+. \tag{15}$$

Then, setting $u \to 0$, we conclude that, for solutions of the IDE problem (5), (10)–(15) the following nonlocal condition must be satisfied

$$cf'(+0) - (\lambda + \lambda_1)f(+0) + \frac{\lambda_1}{n}\int_0^\infty f(s)\exp(-s/n)ds = 0. \tag{16}$$

**Theorem 1.** *A solution of ODE* (9) *that satisfies the conditions* (10)–(15) *is also the solution of IDE* (5) *if and only if the nonlocal condition* (16) *is satisfied.*

**Proof.** The necessity, i.e., the fact that the solution to the IDE problems (5), (10)–(15) satisfies the ODE (9), and the nonlocal condition (16) is obvious due to the construction of the ODE and the reasoning above. Let us prove the sufficiency, namely, that the solution of ODE problem (9), (10)–(15) which satisfies the condition (16), is satisfying also the IDE (5).

Denote by $h$ the left-hand side of the equation (5) where the function f is the solution of ODE problem (9), (10)–(15) which satisfies the condition (16). Then, in view of ODE (9) we have that the function h satisfies the equation

$$h(u) + (m - n)h'(u) - (mn)h''(u) = 0. \tag{17}$$

Let us prove that $h(u) \equiv 0$ using the formulated conditions (10)–(15) and (16). The general solution of the equation (17) has the form

$$h(u) = C_1 e^{u/n} + C_2 e^{-u/m} \tag{18}$$

with arbitrary constants $C_1$, $C_2$. From the conditions (10), (12) we have

$$\lim_{u\to 0} h(u) = cf'(+0) - (\lambda + \lambda_1)f(+0) + \frac{\lambda_1}{n}\int_0^\infty f(s)\exp^{-s/n} ds, \tag{19}$$

then, in view of (16) and (18), the equality

$$C_1 + C_2 = 0 \tag{20}$$

is satisfied. By virtue of conditions (13), (14), it is easy to see that $\lim_{u\to+\infty}(J_{1,n}f)(u) = 1$, and

$$\lim_{u\to\infty} h(u) = -(\lambda + \lambda_1) + \lambda \lim_{u\to\infty}(J_m f)(u) + \lambda_1 = -\lambda + \lambda \lim_{u\to\infty}(J_m f)(u).$$

For $f$ satisfying condition (15), the right-hand side of the last equation is a finite value, therefore, $C_1 = 0$ in (18), and taking equality (20) into account, we obtain that $h \equiv 0$. Thus, $f$ satisfies IDE (5). □

As a result, we have the following

**Corollary 1.** *IDE problem* (5), (10)–(15) *and ODE problem* (9), (10)–(16) *are equivalent.*

## 3. The survival probability as a solution of IDE and ODE problems

To establish a connection between a solution of the IDE problem and the survival probability in the origin problem, we use the so-called sufficiency theorem (for other models, see [4]).

**Theorem 2.** *For the process* (2)*, let the conditions* (**A1**) *and* (**A2**) *be fulfilled with* $a$*,* $b$ *defined in* (3)*. Suppose the IDE* (5) *has a twice continuously differentiable on* $(0,\infty)$ *solution* $f(u)$ *subject to conditions* (11)*,* (13)*, and* (15)*. Then* $f(u)\equiv\varphi(u)$*, i.e.,* $f(u)$ *is the survival probability for the process* (2) *with the initial state* $u$*.*

We omit the proof because it is similar to the proof of Theorem 3.1 in [4]. From Theorem 2 and Corollary 1, we obtain the following obvious.

**Corollary 2.** *For the process* (2)*, let the conditions* (**A1**) *and* (**A2**) *be fulfilled with* $a$*,* $b$ *defined in* (3)*. Suppose there is a twice continuously differentiable on* $(0,\infty)$ *solution* $f(u)$ *to the ODE problem* (9)*,* (10)–(16)*. Then* $f(u)\equiv\varphi(u)$*, i.e.,* $f(u)$ *is the survival probability for the process* (2) *with the initial state* $u$*.*

Proof of the existence of a solution to the ODE problem mentioned above is not considered in this paper. For the case $c=0$, the existence and uniqueness of such a solution to a similar problem is proved in [5]. In the next section, we present an algorithm that allows us to obtain a solution to ODE problems (9), (10)–(16) with nonlocal conditions by solving a boundary value problem for a third-order ODE under the condition that a solution of this problem exists and is integrable at infinity function (the existence of integrable at infinity solution to the third-order ODE follows from the power asymptotics for the derivative of the survival probability satisfying this equation, see [3]).

## 4. Algorithm for solving the ODE problem with nonlocal condition by solving a boundary value problem for ODE with a reduced order

Denote $f'=\psi$ and reformulate the ODE (9) and the conditions (12) and (14) in terms of the function $\psi$. Then we have

$$\frac{b^2}{2}u^2\psi'''(u)+\left[c+(2b^2+a)u+\frac{b^2(n-m)}{2mn}u^2\right]\psi''(u)+$$
$$+\left[b^2+2a-\lambda-\lambda_1+\frac{c(n-m)}{mn}+\frac{(b^2+a)(n-m)}{mn}u-\frac{b^2}{2mn}u^2\right]\psi'(u)+$$
$$+\left[\frac{a(n-m)+\lambda m-\lambda_1 n-c}{mn}-\frac{a}{mn}u\right]\psi(u)=0,\quad u>0, \tag{21}$$
$$\lim_{u\to+0}|\psi(u)|<\infty,\quad \lim_{u\to+0}[u\psi'(u)]=0, \tag{22}$$
$$\lim_{u\to\infty}[u\psi(u)]=0,\quad \lim_{u\to\infty}[u^2\psi'(u)]=0. \tag{23}$$

Since the value of $\psi(+0)=\lim_{u\to+0}|\psi(u)|$ is not defined, we will consider (21)–(23) as a parametric family of problems with the parameter $\psi(+0)=\psi_0$, $0<\psi_0<\infty$. Note that the local condition (16) can be rewritten as

$$\lambda f(0)=c\psi(+0)+\lambda_1\int_0^\infty\psi(s)\exp(-s/n)ds. \tag{24}$$

Let $\widetilde{\psi}$ be an integrable at infinity solution to the ODE problem (21)–(23) with a fixed parameter $\widetilde{\psi}(+0) = \widetilde{\psi}_0 > 0$. Then the solution of the ODE problem (9), (10)–(16) can be obtained through the following steps.

*Step 1*: for the case $c > 0$, to define $\widetilde{f}(0)$ from the condition (24) with function $\widetilde{\psi}$ instead $\psi$; for the case $c < 0$, we set $\widetilde{f}(0) = 0$.

*Step 2*: to calculate the function

$$\widetilde{f}(u) = \widetilde{f}(0) + \int_0^u \widetilde{\psi}(s)ds. \tag{25}$$

*Step 3*: to find $\widetilde{f}(\infty) = \lim_{u\to\infty} \widetilde{f}(u)$ from (25).

*Step 4*: to define the functions $f(u) = \widetilde{f}(u)/\widetilde{f}(\infty)$, $\psi(u) = \widetilde{\psi}(u)/\widetilde{f}(\infty)$.

It is clear that these functions are related by equality $f' = \psi$ and relation (24), and $f$ is the solution of the ODE problem (9), (10)–(16). It is obviously that the unknown value of the parameter $\psi_0 = \psi(+0)$ can be found from the equality $\psi_0 = \widetilde{\psi}_0/\widetilde{f}(\infty)$.

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