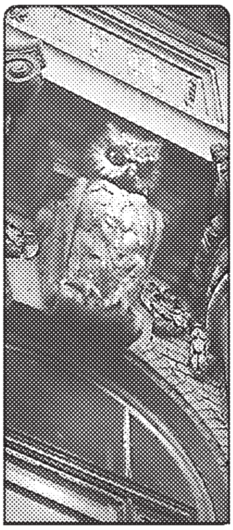

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Article

A highly accurate difference method for solving the Dirichlet problem of the Laplace equation on a rectangular parallelepiped with boundary values in $C^{k,1}$

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Abstract. A three-stage difference method for solving the Dirichlet problem of Laplace's equation on a rectangular parallelepiped is proposed and justified. In the first stage, approximate values of the sum of the pure fourth derivatives of the solution are defined on a cubic grid by the 14-point difference operator. In the second stage, approximate values of the sum of the pure sixth derivatives of the solution are defined on a cubic grid by the simplest 6-point difference operator. In the third stage, the system of difference equations for the sought solution is constructed again by using the 6-point difference operator with the correction by the quantities determined in the first and the second stages. It is proved that the proposed difference solution to the Dirichlet problem converges uniformly with the order $O(h^6(|\ln h|+1))$, when the boundary functions on the faces are from $C^{7,1}$ and on the edges their second, fourth, and sixth derivatives satisfy the compatibility conditions, which follows from the Laplace equation. A numerical experiment is illustrated to support the analysis made.
Keywords: finite difference method, 3D Laplace equation, cubic grids on parallelepiped, 14-point averaging operator, error estimations
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Разностный метод высокой точности при решении задачи Дирихле для уравнения Лапласа на прямоугольном параллелепипеде с граничными значениями в $C^{k,1}$

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Аннотация. В работе предлагается и обосновывается трехэтапный разностный метод для решения задачи Дирихле уравнения Лапласа на прямоугольном параллелепипеде. На первом этапе приближенное значение суммы из чистых четвертых производных решения определяется 14-точечным разностным оператором на кубической сетке. На втором этапе приближенное значение суммы из чистых шестых производных решения определяется простейшим 6-точечным разностным оператором. На третьем этапе система разностных уравнений для искомого решения конструируется также с помощью 6-точечного разностного оператора с коррекцией по результатам первого и второго этапов. Доказано, что предложенная разностная схема решения для задачи Дирихле сходится со скоростью $O(h^6(|\ln h|+1))$, когда граничные функции на гранях из $C^{7,1}$, а на ребрах их вторые, четвертые и шестые производные удовлетворяют условию согласования, вытекающего из уравнения Лапласа.

Ключевые слова: конечно разностный метод, 3D уравнения Лапласа, кубические сетки в параллелепипеде, 14-точечный оператор усреднения, оценки погрешности

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Introduction

A highly accurate method is one of the powerful tools for reducing the number of unknowns, which is the main problem in the numerical solution of differential equations to get reasonable results. In the most of approximations to get highly accurate results the difference operators with a high number of patterns are used, which increase the number of bandwidth of the difference equations. It is obvious that the complexity of the realization methods for the difference equations increases depending on the number of the bandwidth of the matrices of these systems of difference equations. As it was shown by R. E. Tarjan [1], in the case of the Gaussian elimination method the bandwidth elimination for $n \times n$ matrices with the bandwidth b, the computational cost is of order $O(b^2 n)$.

One of the effective methods of increased accuracy which uses the simplest finite difference approximation by correcting the right-hand-side term with the application of the high order differences of the numerical solution of differential equation, was proposed by L. Fox [2] without theoretical justification. Some modification of Fox's approach was given by Woods [3]. A theoretical justification of Fox's method was presented by Volkov in [4, 5]. From Volkov's results in the case of the Dirichlet problem for Poisson's equation on a rectangular domain Π, it follows that the approximate solution obtained by the q-th correction of the right-hand side of the 5-point scheme, the convergence order in the uniform metric is $O(h^{2q})$, h is the mesh step, when the exact solution u has $(2q+2)$-th derivatives on $\overline{\Pi}$ satisfying a Hölder condition with exponent $\lambda \in (0,1)$, i.e., $u \in C^{2q+2,\lambda}(\overline{\Pi})$.

In Berikelashvili and Midodashvili [6] it is proved that the corrected 5-point difference scheme on the rectangular grid is convergent at the rate $O(|h|^m)$, $|h|^2 = h_1^2 + h_2^2$, in the discrete L_2-norm, provided that the exact solution belongs to the Sobolev space W_2^m, $m \in [2,4]$.

In Volkov [7] a two-stage difference method for solving the Dirichlet problem for Laplace's equation on a rectangular parallelepiped was proposed. It was assumed that the given boundary functions on the faces of a parallelepiped have the sixth derivatives satisfying the Hölder condition, and on the edges, besides the continuity they satisfy the compatibility condition for second derivatives, which results from the Laplace equation. It was proved that by using a simple 7-point scheme in two stages the order of uniform error can be improved up to $O(h^4 \ln h^{-1})$. From the conditions imposed on the boundary functions in [7], it does not follow as it was mistakenly declared in [6] that the exact solution belongs to $C^{6,\lambda}(\overline{\Pi})$.

Moreover, as it was shown in [8], the theoretical justification of the difference schemes needs special attention when the boundary values of a solution belong to the Hölder classes $C^{2l-1,1}$ and $2l-2$ order derivatives satisfy the conjunction condition followed from the Laplace equation. In this case, some of $2l$ order derivatives may be unbounded near the boundary of the solution domain, and for the rate of convergence of the 27-point difference solution, when $l=3$, $O(h^6(\|\ln h|+1))$ of order is obtained.

In this paper, a three-stage difference method constructed a special combination of 15-point and 7-point schemes for solving the Dirichlet problem of Laplace's equation on a rectangular parallelepiped is proposed and justified. It is proved that the obtained difference method converges uniformly with an order of $O(h^6(\|\ln h|+1))$ when the boundary functions on the faces are from $C^{7,1}$, and on the edges their second, fourth, and sixth derivatives satisfy the compatibility conditions which follows from the Laplace equation.

A numerical experiment is illustrated to support the analysis made.

1. The Dirichlet problem on rectangular parallelepiped

Let $R = \{(x_1,x_2,x_3):\ 0 < x_i < a_i,\ i=1,2,3\}$ be an open rectangular parallelepiped, Γ_j $(j=1,2,...,6)$ be its faces including the boundaries such that Γ_j for $j=1,2,3$ (for $j=4,5,6$) belongs to the plane $x_j=0$ (to the plane $x_{j-3}=a_{j-3}$). Let $\Gamma = \cup_{j=1}^{6}\Gamma_j$ be the boundary of the parallelepiped, let γ be the union of the edges of R, and let $\Gamma_j' = \Gamma_j \backslash \gamma$ and $\gamma_{\mu\nu} = \Gamma\mu \cap \Gamma\nu$. We say that $f \in C^{k,\lambda}(D)$ if f has continuous $k-$th derivatives on D satisfying a Hölder condition with exponent $\lambda \in (0,1]$, which is a Lipschitz condition when $\lambda = 1$.

We consider the boundary value problem

$$\triangle u = 0 \text{ on } R, \quad u = \varphi_j \text{ on } \Gamma_j, \quad j=1,2,...,6, \tag{1}$$

where $\triangle \equiv \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \partial^2/\partial x_3^2$, φ_j are given functions.

Assume that

$$\varphi_j \in C^{7,1}(\Gamma_j), \quad j=1,2,...,6, \tag{2}$$

$$\varphi_\mu = \varphi_\nu \quad \text{on } \gamma_{\mu\nu}, \tag{3}$$

$$\frac{\partial^2\varphi_\mu}{\partial t_\mu^2} + \frac{\partial^2\varphi_\nu}{\partial t_\nu^2} + \frac{\partial^2\varphi_\mu}{\partial t_{\mu\nu}^2} = 0 \text{ on } \gamma_{\mu\nu}, \tag{4}$$

$$\frac{\partial^4\varphi_\mu}{\partial t_\mu^4} + \frac{\partial^4\varphi_\mu}{\partial t_\mu^2\partial t_{\mu\nu}^2} = \frac{\partial^4\varphi_\nu}{\partial t_\nu^4} + \frac{\partial^4\varphi_\mu}{\partial t_\nu^2\partial t_{\nu\mu}^2} \text{ on } \gamma_{\mu\nu}, \tag{5}$$

$$\frac{\partial^6\varphi_\mu}{\partial t_\mu^6} + \frac{\partial^6\varphi_\mu}{\partial t_\mu^4\partial t_{\mu\nu}^2} + \frac{\partial^6\varphi_\mu}{\partial t_\mu^4\partial t_\nu^2} = \frac{\partial^6\varphi_\nu}{\partial t_\mu^2\partial t_\nu^4} + \frac{\partial^6\varphi_\nu}{\partial t_\nu^6} + \frac{\partial^6\varphi_\mu}{\partial t_\nu^4\partial t_{\mu\nu}^2} \text{ on } \gamma_{\mu\nu}, \tag{6}$$

where $1 \leqslant \mu < \nu \leqslant 6$, $\nu - \mu \neq 3$, $t_{\mu\nu}$ is an element in $\gamma_{\mu\nu}$, t_μ and t_ν is an element of the normal to $\gamma_{\mu\nu}$ on the face Γ_μ and Γ_ν, respectively.

Let $\sigma(j) = 3\{j/3\} + 1$, where $\{a\}$ is the fractional part of a.

Lemma 1. *In the open parallelepiped R it holds that*

$$\frac{\partial^4 u(x_1,x_2,x_3)}{\partial x_j^4} = \frac{\partial^4 u(x_1,x_2,x_3)}{\partial x_{\sigma(j)}^4} + \frac{\partial^4 u(x_1,x_2,x_3)}{\partial x_{\sigma(j+1)}^4} + 2\frac{\partial^4 u(x_1,x_2,x_3)}{\partial x_{\sigma(j)}^2 \partial x_{\sigma(j+1)}^2}, \quad (7)$$

$$\frac{\partial^6 u(x_1,x_2,x_3)}{\partial x_j^6} = -\frac{\partial^6 u(x_1,x_2,x_3)}{\partial x_{\sigma(j)}^6} - \frac{\partial^6 u(x_1,x_2,x_3)}{\partial x_{\sigma(j+1)}^6} - $$
$$-3\frac{\partial^6 u(x_1,x_2,x_3)}{\partial x_{\sigma(j)}^4 \partial x_{\sigma(j+1)}^2} - 3\frac{\partial^6 u(x_1,x_2,x_3)}{\partial x_{\sigma(j)}^2 \partial x_{\sigma(j+1)}^4}, \quad (8)$$

where u is the solution to the Dirichlet problem (1).

Proof. The proof directly follows from the Laplace equation. □

On $\overline{R}$, we define the functions

$$v^k = v^k(x_1,x_2,x_3) = \frac{1}{k}\sum_{j=1}^{3} \frac{\partial^{2k} u(x_1,x_2,x_3)}{\partial x_j^{2k}}, \quad k = 2,3, \quad (9)$$

where u is the solution to the Dirichlet problem (1).

Lemma 2. *The functions* (9) *coincide with the unique continuous solution on $\overline{R}$ of the boundary value problems*

$$\Delta v^k = 0 \text{ on } R, \quad v^k = \psi_j^k \text{ on } \Gamma_j, \quad j = 1,2,...,6, \quad k = 2,3, \quad (10)$$

where

$$\psi_j^2 = \psi_j^2(x_{\sigma(j)}, x_{\sigma(j+1)}) = \frac{\partial^4 \varphi_j(x_{\sigma(j)}, x_{\sigma(j+1)})}{\partial x_{\sigma(j)}^4} + \frac{\partial^4 \varphi_j(x_{\sigma(j)}, x_{\sigma(j+1)})}{\partial x_{\sigma(j+1)}^4} +$$
$$+\frac{\partial^4 \varphi_j(x_{\sigma(j)}, x_{\sigma(j+1)})}{\partial x_{\sigma(j)}^2 \partial x_{\sigma(j+1)}^2}, \quad (11)$$

$$\psi_j^3 = \psi_j^3(x_{\sigma(j)}, x_{\sigma(j+1)}) = -\frac{\partial^6 \varphi_j(x_{\sigma(j)}, x_{\sigma(j+1)})}{\partial x_{\sigma(j)}^4 \partial x_{\sigma(j+1)}^2} - \frac{\partial^6 \varphi_j(x_{\sigma(j)}, x_{\sigma(j+1)})}{\partial x_{\sigma(j)}^2 \partial x_{\sigma(j+1)}^4}. \quad (12)$$

Proof. On the basis of (2)–(6), Theorem 2.1 in [9] it follows that a solution u of problem (1) belongs to the class $C^{7,\lambda}(\overline{R})$, $0 < \lambda < 1$. Since any order derivatives of a harmonic function are also harmonic, the functions v^k, $k = 2,3$ satisfy Laplace's equation. The boundary conditions in (10) with (11) and (12) follow from (1), Lemma 1 and (9). Then by Theorem 3.1 in [9] each of the functions v^k, $k = 2,3$ is the unique continuous solution on $\overline{R}$ of problem (10). □

Lemma 3. *Even order derivatives in the form*

$$\frac{\partial^8 u}{\partial x^{2p} \partial x^{2q} \partial x^{8-2p-2q}}, \quad 0 \leqslant p \leqslant 4, \quad 0 \leqslant q \leqslant 4-p, \quad (13)$$

of the solution u of problem (1) *are bounded on R.*

Proof. Let $\omega = \frac{\partial^6 u}{\partial x_1^6}$. We have

$$\Delta\omega = 0 \text{ on } R, \quad \omega = \Phi_j \quad \text{on } \Gamma_j, \quad j = 1, 2, ..., 6,$$

where

$$\Phi_j = \frac{\partial^6 \varphi_j}{\partial x_1^6}, \quad j = 2, 3, 5, 6, \tag{14}$$

$$\Phi_j = -\frac{\partial^6 \varphi_j}{\partial x_2^6} - 3\frac{\partial^6 \varphi_j}{\partial x_2^4 \partial x_3^2} - 3\frac{\partial^6 \varphi_j}{\partial x_2^2 \partial x_3^4} - \frac{\partial^6 \varphi_j}{\partial x_3^6}, \quad j = 1, 4. \tag{15}$$

From (1)–(6) follows that the boundary functions Φ_j, $j = 1, 2, ..., 6$ defined by (14) and (15) satisfy the conditions

$$\Phi_j \in C^{1,1}(\Gamma_j), \quad \Phi_\mu = \Phi_\nu \quad \text{on } \gamma_{\mu\nu}.$$

Then, on the basis of Theorem 4.1 in [9] the pure second-order derivatives of the function ω are bounded in R. Then

$$\sup_{(x_1,x_2,x_3)\in R} \left|\frac{\partial^8 u}{\partial x_1^8}\right| = \sup_{(x_1,x_2,x_3)\in R} \left|\frac{\partial^2 \omega}{\partial x_1^2}\right| < \infty,$$

$$\sup_{(x_1,x_2,x_3)\in R} \left|\frac{\partial^8 u}{\partial x_1^6 \partial x_2^2}\right| = \sup_{(x_1,x_2,x_3)\in R} \left|\frac{\partial^2 \omega}{\partial x_2^2}\right| < \infty,$$

$$\sup_{(x_1,x_2,x_3)\in R} \left|\frac{\partial^8 u}{\partial x_1^6 \partial x_3^2}\right| = \sup_{(x_1,x_2,x_3)\in R} \left|\frac{\partial^2 \omega}{\partial x_3^2}\right| < \infty.$$

Similarly, by taking $\omega = \frac{\partial^6 u}{\partial x_2^6}$, and $\omega = \frac{\partial^6 u}{\partial x_3^6}$ the boundedness of the remainder even order derivatives in (13) are proved. □

Lemma 4. *Let u be the solution of problem (1), $\rho(x_1, x_2, x_3)$ be the distance from the current point of R to its boundary and let $\partial/\partial l \equiv \alpha_1\partial/\partial x_1 + \alpha_2\partial/\partial x_2 + \alpha_3\partial/\partial x_3$, $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$ be the l — directional differentiation operator. Then*

$$\left|\frac{\partial^{10} u(x_1, x_2, x_3)}{\partial l^{10}}\right| \leqslant c_0 \rho^{-2}(x_1, x_2, x_3), \quad (x_1, x_2, x_3) \in R, \tag{16}$$

where c_0 is a constant independent of the direction l of the operator $\partial/\partial l$.

Proof. Since any tenth-order derivative of u can be obtained by two times differentiating some of the derivatives of the form (13), on the basis of Lemma 3 from [10, Chapter 4, Sec. 3] and Lemma 3, we have

$$\max_{0\leqslant\mu\leqslant 10} \max_{0\leqslant\nu\leqslant 10-\mu} \left|\frac{\partial^{10} u(x_1, x_2, x_3)}{\partial x_1^\mu \partial x_2^\nu \partial x_3^{10-\mu-\nu}}\right| \leqslant c_1 \rho^{-2}(x_1, x_2, x_3), \quad (x_1, x_2, x_3) \in R. \tag{17}$$

From (17) follows the inequality (16). □

2. $O(h^6 |\ln h|)$ order accurate approximate solution

Consider a cubic mesh with the mesh size $h > 0$ formed by the planes $x_i = 0, h, 2h, ...(i = 1, 2, 3)$. Assume that $a_i/h \geqslant 4$ $(i = 1, 2, 3)$ are integers. Let D_h be the set of mesh nodes, $R_h = R \cap D_h$, $\Gamma_{jh} = \Gamma_j \cap D_h$, $\Gamma_h = \Gamma \cap D_h$, $\Gamma'_{jh} = \Gamma'_j \cap D_h$, and $\Gamma'_h = \Gamma'_{1h} \cup ... \cup \Gamma'_{6h}$. We put $\overline{R}_h = R_h \cup \Gamma_h$, $\overline{R}'_h = R_h \cup \Gamma'_h$. Let $R_h^k \subset R_h$ be the set of nodes of R_h lying at a distance of kh away from the boundary Γ of R. It is clear that $k = 1, 2, ..., N(h)$, where $N(h) = [\min\{a_1, a_2, a_3\}/(2h)]$.

For the grid functions on R_h, we consider the 6-point difference operator A as

$$Au(x_1, x_2, x_3) = \frac{1}{6} \sum_{p=1^{(1)}}^{6} u_p,$$

and the 14-point difference operator S as

$$Su(x_1, x_2, x_3) \equiv \frac{1}{56} \left(8 \sum_{p=1^{(1)}}^{6} u_p + \sum_{q=7^{(3)}}^{14} u_q \right),$$

where the sum $\sum_{(k)}$ is taken over the grid nodes that are at a distance of $\sqrt{k}h$ from the point (x_1, x_2, x_3), u_p and u_q are the values of u at the corresponding grid points.

Consider two systems of grid equations

$$v_h = Av_h + g_h, \quad \text{on } R_h, \quad v_h = 0 \quad \text{on } \Gamma'_h, \tag{18}$$

$$\overline{v}_h = A\overline{v}_h + \overline{g}_h, \quad \text{on } R_h, \quad \overline{v}_h = 0 \quad \text{on } \Gamma'_h, \tag{19}$$

where g_h and $\overline{g}_h$ are given functions and $|g_h| \leqslant \overline{g}_h$ on R_h.

Lemma 5. *The solutions v_h and $\overline{v}_h$ to systems (18) and (19) satisfy the inequality*

$$|v_h| \leqslant \overline{v}_h \quad \text{on } R_h.$$

Proof. The proof of Lemma 5 follows from the comparison theorem (see [11, Chapter 4]). □

2.1. The first stage

Let v_h^2 be a solution of the following finite difference problem

$$v_h^2 = Sv_h^2 \text{ on } R_h, \quad v_h^2 = \psi_j \text{ on } \Gamma_{jh}, \quad j = 1, 2, ..., 6, \tag{20}$$

where ψ_j, $j = 1, 2, ..., 6$ are functions defined in (11).

Let $c, c_1, c_2, ...$ denote positive constants independent of the nearby multiplier, of which some possibly have identical values.

Lemma 6. *The following estimation holds*

$$\max_{(x_1,x_2,x_3)\in\overline{R}_h} |v_h^2 - v^2| \leqslant c_1 h^4(|\ln h| + 1),$$

where v^2 is the function (9) when $k = 2$ and v_h^2 is the solution of the system of grid equations (20).

Proof. By Lemma 2,

$$\Delta v^2 = 0 \text{ on } R, \quad v = \psi_j^2 \text{ on } \Gamma_j, \quad j = 1, 2, ..., 6,$$

where functions ψ_j^2 defined by (11).

For the error function

$$\varepsilon_h^2 = v_h^2 - v^2, \tag{21}$$

we have

$$\varepsilon_h^2 = S\varepsilon_h^2 + \left(Sv^2 - v^2\right) \text{ on} R_h, \quad \varepsilon_h^2 = 0 \quad \text{on } \Gamma_h.$$

Let ε_h^2 be represented as

$$\varepsilon_h^2 = \varepsilon_h^{2,1} + \varepsilon_h^{2,2} + ... + \varepsilon_h^{2,N(h)}, \tag{22}$$

where $\varepsilon_h^{2,k}$, $1 \leqslant k \leqslant N(h)$ is the solution of system

$$\varepsilon_h^{2,k} = S\varepsilon_h^{2,k} + \xi^k \ \text{ on } R_h, \quad \varepsilon_h^{2,k} = 0 \ \text{ on } \Gamma_h,$$

with

$$\xi^k = \begin{cases} Sv^2 - v^2 & \text{on } R_h^k, \\ 0 & \text{on } R_h \backslash R_h^k. \end{cases}$$

By virtue of Lemma 4 in [12], we have

$$\max_{(x_1,x_2,x_3)\in R_h} |\varepsilon_h^{2,k}| \leqslant 5k \max_{(x_1,x_2,x_3)\in R_h^k} |Sv^2 - v^2|, \quad 1 \leqslant k \leqslant N(h). \tag{23}$$

To estimate $Sv^2 - v^2$ on R_h^k, for $k = 1, 2, ..., N(h)$, first we note that, from (9) and Lemma 4 follows

$$\left|\frac{\partial^6 v^2(x_1,x_2,x_3)}{\partial l^6}\right| \leqslant c_2 \left|\frac{\partial^{10} u(x_1,x_2,x_3)}{\partial l^{10}}\right| \leqslant c_3 \rho^{-2}(x_1,x_2,x_3), \quad (x_1,x_2,x_3) \in R. \tag{24}$$

Let $x_0 = (x_{10}, x_{20}, x_{30})$ be a node of the grid $R_h^{k_0} \subset R_h$, where k_0 be an arbitrary integer number $2 \leqslant k_0 \leqslant N(h)$ and let $r_6(x_1,x_2,x_3;x_0)$ be the Lagrange remainder corresponding to this point in Taylor formula

$$v^2(x_1,x_2,x_3) = p_5(x_1,x_2,x_3;x_0) + r_6(x_1,x_2,x_3;x_0), \tag{25}$$

where

$$Sp_5(x_{10},x_{20},x_{30};x_0) = v^2(x_{10},x_{20},x_{30}). \tag{26}$$

Then on the basis of (24), we have

$$Sr_6(x_{10},x_{20},x_{30};x_0) \leqslant c_4 \frac{h^6}{(k_0 h)^2} = c_4 \frac{h^4}{{k_0}^2}. \tag{27}$$

From (25)–(27), we obtain

$$\max_{(x_1,x_2,x_3)\in R_h^k} |Sv^2 - v^2| \leqslant c_4 \frac{h^4}{k^2}, \quad 2 \leqslant k \leqslant N(h). \tag{28}$$

Let $x_0 = (x_{10}, x_{20}, x_{30})$ be a node of the grid $R_h^1 \subset R_h$, and the nodes of operator S lie at the distance h or $\sqrt{3}h$ from this point. We estimate r_6 at the nodes of the operator S. To do this we take a node $(x_{10} - h, x_{20} - h, x_{30} + h)$ and consider the continuous function

$$\widetilde{v}^2(s) = v^2\left(x_{10} - \frac{s}{\sqrt{3}}, x_{20} - \frac{s}{\sqrt{3}}, x_{30} + \frac{s}{\sqrt{3}}\right), \quad -\sqrt{3}h \leqslant s \leqslant \sqrt{3}h, \tag{29}$$

of one variable s. By estimation (24), we have

$$\left|\frac{d^6}{ds^6}\widetilde{v}^2(s)\right| \leqslant c_5 \left(\sqrt{3}h - s\right)^{-2}, \quad 0 \leqslant s < \sqrt{3}h. \tag{30}$$

The function

$$\widetilde{r}_6 = r_6\left(x_{10} - \frac{s}{\sqrt{3}}, x_{20} - \frac{s}{\sqrt{3}}, x_{30} + \frac{s}{\sqrt{3}}; x_0\right)$$

is the remainder term of the representation of the function (29) around the point $s = 0$ by Taylor's formula with the fifth order polynomial.

By using integral form of the remainder term and (30), we obtain (see [8])

$$\left| r_6\left(x_{10}-\frac{s}{\sqrt{3}}, x_{20}-\frac{s}{\sqrt{3}}, x_{30}+\frac{s}{\sqrt{3}}; x_0\right)\right| \leqslant c_6 h^4. \tag{31}$$

For the remaining nodes of the operator S the estimation (31) can be obtained analogically. Since the maximum norm of the operator S is equal to one, we have

$$|Sr_6(x_{10}, x_{20}, x_{30}; x_0)| \leqslant c_7 h^4. \tag{32}$$

By (25), (26) and (32), we obtain

$$\max_{(x_1,x_2,x_3)\in R_h^1} |Sv^2 - v^2| \leqslant c_8 h^4. \tag{33}$$

On the basis of (21)–(23), (28) and (33), we have

$$\max_{(x_1,x_2,x_3)\in R_h} |v_h^2 - v^2| \leqslant c_9 h^4 \sum_{k=1}^{N(h)} \frac{1}{k} \leqslant c_1 h^4 \left(\|\ln h| + 1\right).$$

□

2.2. The second stage

Let v_h^3 be a solution to the following finite difference problem

$$v_h^3 = Av_h^3 \text{ on } R_h, \quad v_h^3 = \psi_j^3 \text{ on } \Gamma'_{jh}, \quad j = 1, 2, ..., 6, \tag{34}$$

where ψ_j^3, $j = 1, 2, ..., 6$ are functions defined by (12).

Lemma 7. *On R_h, it holds that,*

$$\max_{(x_1,x_2,x_3)\in \overline{R}'_h} |v_h^3 - v^3| \leqslant c_2 h^2(|\ln h| + 1), \tag{35}$$

where v^3 is the function (9) for $k = 3$, v_h^3 is a solution to system (34).

Proof. By Lemma 2, we have

$$\Delta v^3 = 0 \text{ on } R, \quad v^3 = \psi_j^3 \text{ on } \Gamma_j, \quad j = 1, 2, ..., 6, \tag{36}$$

where functions ψ_j^3 define boundary values in (12), and from (2)–(6) it follows that

$$\psi_j^3 \in C^{1,1}(\Gamma_j), \quad 0 < \lambda < 1, \quad j = 1, 2, ..., 6, \tag{37}$$

$$\psi_\mu^3 = \psi_\nu^3 \text{ on } \gamma_{\mu\nu}, \tag{38}$$

on the basis of (36)–(38) that satisfy the conditions of Theorem 5.1 in [9] which follows estimation (35). □

2.3. The third stage

Let v_h^2 and v_h^3 be the solution of the difference problems (20)) and (34) respectively. We approximate the solution of the given Dirichlet problem (1) on the grid $\overline{R}_h$ as a solution u_h of the following difference problem

$$u_h = Au_h - \frac{h^4}{36}v_h^2 - \frac{h^6}{720}v_h^3 \text{ on } R_h, \tag{39}$$

$$u_h = \varphi_j \text{ on } \Gamma'_{jh}, \quad j = 1, 2, ..., 6. \tag{40}$$

Theorem 1. *Under the conditions* (2)–(6)*, the estimation*

$$\max_{(x_1,x_2,x_3)\in\overline{R}'_h} |u_h - u| \leqslant c_3 h^6(\|\ln h| + 1), \tag{41}$$

is valid, where u *is the solution of the Dirichlet problem* (1) u_h *is the solution of system* (39)*,* (40).

Proof. Under the smoothness properties of the boundary values specified in (2)–(6), the solution u of the Dirichlet problem (1) has eighth-order partial derivatives that are continuous on R, and by using Taylor's formula with the remainder term in the Lagrange form for each $(x_1, x_2, x_3) \in R_h$, we obtain

$$u(x_1, x_2, x_3) = Au(x_1, x_2, x_3) - \frac{h^4}{36}v^2 - \frac{h^6}{720}v^3 - r(x_1, x_2, x_3), \tag{42}$$

where v^k, $k = 2, 3$ are the functions defined by (9)

$$\max_{(x_1,x_2,x_3)\in R} |r(x_1, x_2, x_3)| \leqslant c_4 h^8. \tag{43}$$

We put

$$\varepsilon_h = u_h - u \quad \text{on } \overline{R}'_h,$$

where u_h is the solution of the finite difference problem (39), (40).

From (39) and (42), and taking into account that $u_h = u = \varphi_j$ on Γ_{jh}, we obtain the following system of difference equations for the error ε_h:

$$\varepsilon_h = A\varepsilon_h + \frac{h^4}{36}(v^2 - v_h^2) + \frac{h^6}{720}(v^3 - v_h^3) + r \quad \text{on } R_h, \tag{44}$$

$$\varepsilon_h = 0 \quad \text{on } \Gamma'_h. \tag{45}$$

On the basis of Lemma 6 , Lemma 7, and the estimation (43)), we obtain

$$\left|\frac{h^4}{36}(v^2 - v_h^2) + \frac{h^6}{720}(v^3 - v_h^3) + r\right| \leqslant c_5 h^8(\|\ln h\| + 1),$$

where $c_5 = \max\{c_1/36, c_2/720, c_4\}$.

Furthermore, from Lemma 5 it follows that for the solution ε_h of problem (44), (45) the following estimation is true

$$|\varepsilon_h| \leqslant \overline{\varepsilon}_h, \tag{46}$$

where $\overline{\varepsilon}_h$ is a solution of the problem

$$\overline{\varepsilon}_h = A\overline{\varepsilon}_h + c_5 h^8(|\ln h| + 1) \quad \text{on } R_h, \quad \overline{\varepsilon}_h \geqslant 0 \quad \text{on } \Gamma'_h. \tag{47}$$

It is easy to check that the function $\overline{\varepsilon}_h = c_5 h^6(\|\ln h| + 1)(l^2 - r^2)$, where $l = \sqrt{a_1^2 + a_2^2 + a_3^2}$, and $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$ is a solution of problem (47). Then from (46), follows (41). □

3. Numerical results

Let $R = \{(x_1, x_2, x_3) : 0 < x_i < 1, \ i = 1, 2, 3\}$, and let Γ_j, $j = 1, 2, ..., 6$ be its faces.

$$\Delta u = 0 \quad \text{on } R, \quad u = \varphi(x_1, x_2, x_3) \quad \text{on } \Gamma, \tag{48}$$

where

$$\varphi(x_1, x_2, x_3) = e^{3x_1}\cosh(4x_2)\cos(5x_3) + \left(x_1^8 - 28x_1^6x_2^2 + 70x_1^4x_2^4 - 28x_1^2x_2^6 + x_2^8\right)\tan^{-1}\left(\frac{x_2}{x_1}\right) +$$

$$+\left(8x_1^7x_2-56x_1^5x_2^3+56x_1^3x_2^5-8x_1x_2^7\right)\ln\sqrt{x_1^2+x_2^2},$$

is the exact solution of problem (48) and $\varphi\in C^{7,1}(\Gamma)$.

We use the following notations:

$$\|U_h-U\|_{\Omega_h}=\max_{\Omega_h}|U_h-U|,\quad E_m=\frac{\|U_h-U_{2^{-m}}\|_{\Omega_h}}{\|U_h-U_{2^{-(m+1)}}\|_{\Omega_h}},$$

where U is the trace of the exact solution of the continuous problem on Ω_h, and U_h is its approximate value obtained by the proposed method.

The numerical results given in Table show that the maximum error of the approximate solution obtained by the proposed method absolute values convergent of order $O(h^6\ln h)$, since $2^6>E_n>2^6n/(n+1)$.

Table. Numerical results for Problem (48)

$h=2^{-n}$	$\max_{\Pi^h}\lvert u_{2^{-n}}-u\rvert$	E_n	$2^6n/(n+1)$
2^{-3}	$1.537D-07$	48.394	48.000
2^{-4}	$3.176D-09$	60.231	51.200
2^{-5}	$5.273D-11$	62.625	53.333
2^{-6}	$8.420D-13$	63.071	54.857
2^{-7}	$1.335D-14$	–	–

Conclusion

A new three-stage difference method with an accuracy of order $O(h^6(|\ln h|+1))$, where h is mesh size, is proposed and justified by using one fourth-order and two second-order schemes for the approximate solution of the 3D Laplace's equation. It is assumed that the boundary functions on the faces are from $C^{7,1}$, and on the edges, their second, fourth, and sixth derivatives satisfy the compatibility conditions, which follows from the Laplace equation.

The idea of this method can be used to design a new scheme with an order of convergence $O(h^8(|\ln h|+1))$, when $\varphi_j\in C^{9,1}(\Gamma_j)$, $j=1,\dots,6$.

Moreover, from the estimation (41) the multiplier $|\ln h|$ can be removed by replacing in (2) the condition $\varphi_j\in C^{7,1}(\Gamma_j)$ with the condition $\varphi_j\in C^{8,\lambda}(\Gamma_j)$, $0<\lambda<1$.

The proposed method can be applied when parallelepiped is used as one of the covering figures in some version of domain decomposition methods [13], in the composite grids method for problems in polyhedra and a prism with polygonal base (see [14, 15]). Furthermore, this method can be used to highly approximate the derivatives of the unknown solution of Laplace's equation (see [16–19]).

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