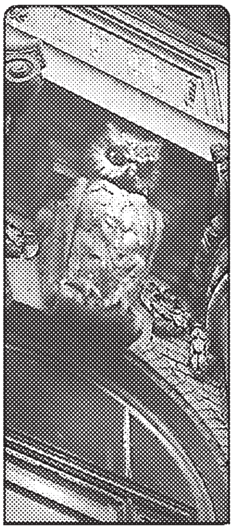

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Article

On semigroups of relations with the operation of the rectangular product

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Abstract. A set of binary relations closed with respect to some collection of operations on relations forms an algebra called an algebra of relations. The theory of algebras of relations is an essential part of modern algebraic logic and has numerous applications in semigroup theory. The following problems naturally arise when classes of algebras of relation are considered: find a system of axioms for these classes, and find a basis of of identities (quasi-identities) for the varieties (quasi-varieties) generated by these classes. In the paper, these problems are solved for the class of semigroups of relation with the binary associative operation of the rectangular product, the result of which is the Cartesian product of the first projection of the first relation on the second projection of the second one.
Keywords: algebra of relations, primitive positive operation, variety, quasi-variety, semigroup, partially ordered semigroup
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О полугруппах отношений с операцией прямоугольного произведения

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Аннотация. Множество бинарных отношений, замкнутое относительно некоторой совокупности операций над отношениями, образует алгебру, называемую алгеброй отношений. Теория алгебры отношений является существенной частью современной алгебраической логики и имеет многочисленные приложения в теории полугрупп. При рассмотрении классов алгебры отношений естественно возникают следующие проблемы: найти систему аксиом для этих классов, найти базис тождеств (квазитождеств) для многообразий (квазимногообразий), порожденных этими классами. В статье обозначенные проблемы решаются для класса полугрупп отношений с бинарной ассоциативной операцией прямоугольного произведения, результатом которой является декартово произведение первой проекции первого отношения на вторую проекцию второго.

Ключевые слова: алгебра отношений, примитивно-позитивная операция, многообразие, квазимногообразие, полугруппа, частично упорядоченная полугруппа

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Introduction

Let $\mathrm{Rel}(U)$ be the set of all binary relations on a base set U. A set of binary relations $\Phi \subseteq \mathrm{Rel}(U)$ closed with respect to some collection Ω of operations on relations forms an algebra (Φ, Ω) called an *algebra of relations*. The theory of algebras of relations is an essential part of modern algebraic logic and has numerous applications in the theory of semigroups (see the remarkable survey [1]).

Denote by $\mathrm{R}\{\Omega\}$ the class of all algebras isomorphic to the ones whose elements are binary relations and whose operations are members of Ω. Let $\mathrm{V}\{\Omega\}$ be the variety and let $\mathrm{Q}\{\Omega\}$ be the quasi-variety generated by $\mathrm{R}\{\Omega\}$.

The following problems naturally arise when the class $\mathrm{R}\{\Omega\}$ is considered.

1. *Find a system of axioms for the class* $\mathrm{R}\{\Omega\}$.
2. *Is the class* $\mathrm{R}\{\Omega\}$ *finitely axiomatizable?*
3. *Find a basis of quasi-identities for the quasi-variety* $\mathrm{Q}\{\Omega\}$.
4. *Is the quasi-variety* $\mathrm{Q}\{\Omega\}$ *finitely based?*
4. *Find a basis of identities for the variety* $\mathrm{V}\{\Omega\}$.
6. *Is the variety* $\mathrm{V}\{\Omega\}$ *finitely based?*
7. *Does the class* $\mathrm{R}\{\Omega\}$ *form a quasi-variety?*
8. *Does the quasi-variety* $\mathrm{Q}\{\Omega\}$ *form a variety?*

Numerous studies have been devoted to solving these problems for various classes of algebras of relations. The first mathematician who treated algebras of relations from the point of view of universal algebra was A. Tarski [2]. He considered algebras of relations (Tarski's algebras of relations) with the following operations: Boolean operations $\cup, \cap, ^{-}$; operations of relational product $\circ$ and relational inverse $^{-1}$; constant operations Δ (diagonal relation), $\varnothing$ (empty relation), $\nabla = U \times U$ (universal relation). He showed that the class $\mathrm{R}\{\circ, ^{-1}, \cup, \cap, ^{-}, \Delta, \varnothing, \nabla\}$ is not a quasi-variety and the quasi-variety generated by this class forms a variety [3]. R. Lyndon [4] found the infinite base of this variety and D. Monk [5] showed that it is not finitely based.

A little later, B. Jónsson considered the class $\mathrm{R}\{\circ, ^{-1}, \cap, \Delta\}$, proved that it forms a quasi-variety, found its infinite base of quasi-identities, and posed problems 4 and 8 for this class [6]. The negative solutions to these problems were obtained in [7] and [8] respectively.

Operations on relations are usually determined using first-order predicate calculus formulas.

Such operations are called *logical*. It is known that classes of algebras of relations with logical operations are axiomatizable [1]. One of the most important classes of logical operations on relations is the class of *primitive-positive* operations [9] (in other terminology — Diophantine operations [10]). An operation on relations is called primitive-positive if it can be defined by a formula of the first-order predicate calculus containing in its prenex normal form only existential quantifiers and conjunctions. Note that the set-theoretical inclusion $\subseteq$ is compatible with all primitive-positive operations. Thus, any algebra of relations with primitive-positive operations (Φ, Ω) can be considered as partially ordered $(\Phi, \Omega, \subseteq)$. The corresponding abstract class of partially ordered algebras will be denoted by $\mathrm{R}\{\Omega, \subseteq\}$. The variety and the quasi-variety generated by the class $\mathrm{R}\{\Omega, \subseteq\}$ will be denoted by $\mathrm{V}\{\Omega, \subseteq\}$ and $\mathrm{Q}\{\Omega, \subseteq\}$ respectively. Problems 1–8 for the class $\mathrm{R}\{\Omega, \subseteq\}$ are formulated in the same way.

One of the most important associative primitive-positive operations is the operation of relational product $\circ$ that is defined as follows:

$$\rho \circ \sigma = \{(u, v) : \ (\exists\ t)(u, t) \in \rho \wedge (t, v) \in \sigma\}.$$

It is well known that the class $\mathrm{R}\{\circ\}$ coincides with the class of all semigroups and the class $\mathrm{R}\{\circ, \subseteq\}$ coincides with the class of all partially ordered semigroups. There are many other binary primitive-positive operations on relations (see, for example [11]). It is interesting to consider problems 1–8 for algebras of relations with these operations. The paper provides a solution to these problems for the class of semigroups of relations with the operation of the rectangular product.

1. Main results

We concentrate our attention on the following primitive-positive operation:

$$\rho * \sigma = \{(u, v) : \ (\exists t, w)(u, t) \in \rho \wedge (w, v) \in \sigma\}.$$

It is easy to see that this operation is associative. Note also that $\rho * \sigma = pr_1\rho \times pr_2\sigma$, where $pr_1\sigma = \{u : \ (\exists t)(u, t) \in \rho\}$ is the first projection of ρ and $pr_2\sigma = \{v : \ (\exists w)(w, v) \in \sigma\}$ is the second projection of σ. Since $pr_1\rho \times pr_2\sigma$ is a rectangular relation, we will treat this operation as the operation of the *rectangular product*.

A *partially ordered semigroup* is an algebraic system $(A, \cdot, \leqslant)$, where $(A, \cdot)$ is a semigroup and $\leqslant$ is a partial order relation on A that is compatible with multiplication, i.e., $x \leqslant y$ implies $xz \leqslant yz$ and $zx \leqslant zy$ for all $x, y, z \in A$.

The main results are formulated in the following theorem and corollaries. Their proofs are based on the description of quasi-equational theories of algebras of relations with primitive-positive operations [10].

Theorem. *The quasi-variety* $\mathrm{Q}\{*, \subseteq\}$ *forms a variety in the class of all partially ordered semigroups. A partially ordered semigroup* $(A, \cdot, \leqslant)$ *belongs to the quasi-variety* $\mathrm{Q}\{*, \subseteq\}$ *if and only if it satisfies the identities:*

$$x \leqslant x^2, \tag{1}$$

$$xyz \leqslant xz. \tag{2}$$

Corollary 1. *The quasi-variety* $\mathrm{Q}\{*\}$ *forms a variety. A semigroup* $(A, \cdot)$ *belongs to the quasi-variety* $\mathrm{Q}\{*\}$ *if and only if it satisfies the identities:*

$$(xy)^2 = xy, \tag{3}$$

$$xyz = xyxz, \tag{4}$$

$$xyzx = xzyx. \tag{5}$$

Corollary 2. *The class* $\mathrm{R}\{*, \subseteq\}$ *does not form a quasi-variety. For a partially ordered semigroup* $(A, \cdot, \leqslant)$, *the following three conditions are equivalent.*

1. $(A, \cdot, \leqslant)$ *belongs to the class* $\mathrm{R}\{*, \subseteq\}$.
2. *One of the following conditions holds:*

   a) $(A, \cdot, \leqslant)$ *satisfies identity* (1) *and the identity*

$$xyz = xz; \tag{6}$$

   b) $(A, \cdot, \leqslant)$ *contains the zero element* o, *satisfies identity* (1) *and the axioms:*

$$y \neq o \Rightarrow xyz = xz, \tag{7}$$
$$o \leqslant x. \tag{8}$$

3. $(A, \cdot, \leqslant)$ *satisfies identity* (1) *and the axioms:*

$$xyz = xy \vee yw = wy = y, \tag{9}$$
$$xy = yx = x \Rightarrow x \leqslant z. \tag{10}$$

Corollary 3. *The class* $\mathrm{R}\{*\}$ *does not form a quasi-variety. For a semigroup* $(A, \cdot)$ *the following three conditions are equivalent.*

1. $(A, \cdot)$ *belongs to the class* $\mathrm{R}\{*\}$.
2. *One of the following conditions holds:*

   a) $(A, \cdot)$ *satisfies identity* (6);

   b) $(A, \cdot)$ *contains the zero element* o *and satisfies axiom* (7).
3. $(A, \cdot)$ *satisfies axiom* (9).

Note also that if the semigroup $(\Phi, \circ)$ of rectangular relations does not contain a zero element, then the operations $\circ$ and $*$ coincide. It follows that the equivalence conditions 1 and 2a of Corollary 3 can be obtained from the results of [12].

Note that the next problem is still open.

Problem. *Let* $(A, \cdot, \leqslant)$ [*respectively,* $(A, \cdot)$] *be a partially ordered semigroup* [*respectively, a semigroup*] *such that the conditions of Corollary* 2 [*respectively, Corollary* 3] *hold, and the set* A *is finite. Is it possible to isomorphically represent* $(A, \cdot, \leqslant)$ [*respectively,* $(A, \cdot)$] *as a partially ordered semigroup* $(\Phi, *, \subseteq)$ [*respectively, as a semigroup* $(\Phi, *)$] *of relations on an appropriate finite set* U.

2. Proofs of results

Step 1. Let us consider the relationship between considered identities and axioms.

Lemma 1. *Identities* (1), (2) *imply identities* (3)–(5). *Identity* (6) *implies identities* (2)–(5). *Identities* (3)–(5) *imply the following three identities:*

$$xyzt = xzyt, \tag{11}$$
$$xy = xy^2, \tag{12}$$
$$xy = x^2y. \tag{13}$$

Proof. First of all, we show that identities (1), (2) imply identities (3)–(5). Indeed, $xy \overset{(1)}{\leqslant} q(xy)^2$ and $(xy)^2 = xyxy \overset{(2)}{\leqslant} xy$, i.e., $(xy)^2 = xy$ (3). Further, $xyxz \overset{(2)}{\leqslant} xyz$ and $xyz \overset{(1)}{\leqslant} (xyz)^2 = xyzxyz \overset{(2)}{\leqslant} xyxz$, i.e., $xyz = xyxz$ (4). Since $xyzx \overset{(1)}{\leqslant} (xyzx)^2 = xyzxxyzx \overset{(2)}{\leqslant}$ $\leqslant xzyx$ and $xzyx \overset{(1)}{\leqslant} (xzyx)^2 = xzyxxzyx \overset{(2)}{\leqslant} xyzx$, we have $xyzx = xzyx$ (5). It is clear that identity (6) implies identities (2)–(5). Let us show that identities (3)–(5) imply identities (11)–(13). Indeed, $xyzt \overset{(4)}{=} xyzxt \overset{(5)}{=} xzyxt \overset{(4)}{=} xzyt$ (11); $xy \overset{(3)}{=} (xy)^2 = xyxy \overset{(4)}{=} xyy = xy^2$ (12); $xy \overset{(3)}{=} (xy)^2 = xyxy \overset{(11)}{=} xxyy = x^2y^2 \overset{(12)}{=} x^2y$ (13). □

Lemma 2. *Conditions* 2 *and* 3 *of Corollaries 2 and 3 are equivalent.*

Proof. Note that we can represent axiom (6) as $(\neg(\forall w)\, yw = wy = y) \Rightarrow xyz = xz$. It follows that this axiom is equivalent to identity (6), if A does not contain a zero element, and it is equivalent to axiom (7) otherwise. Axiom (10) just expresses axiom (8) as a universal formula of the first order language. □

Let us consider the set $\Lambda = \{x_1, \ldots, x_n, \ldots\}$ of individual variables that are interpreted as elements of a semigroup. A semigroup term p is a word in the alphabet Λ, i.e., an expression of the form $x_{j_1}x_{j_2}\ldots x_{j_{m-1}}x_{j_m}$. For convenience, we will also use other letters of the Latin alphabet as variables.

Suppose that $p = x_{j_1}x_{j_2}\ldots x_{j_{m-1}}x_{j_m}$ be the term of a semigroup that satisfies identities (11)–(13). Then without loss of generality, we can assume that all variables $x_{j_2}, \ldots, x_{j_{m-1}}$ are different and different from variables x_{j_1}, x_{j_m}. Moreover, we can also presume that variables $x_{j_2}, \ldots, x_{j_{m-1}}$ can be arbitrarily permuted. In what follows, we will use these properties without special mentions.

Step 2. Let $(\Phi, *, \subseteq)$ be the partially ordered semigroup of relations with the operation of the rectangular product. Since $\rho * \sigma = pr_1\rho \times pr_2\sigma$, we have $\rho \subseteq pr_1\rho \times pr_2\rho = \rho * \rho$, i.e., identity (1) holds. Note that $\rho * \pi * \sigma = pr_1\rho \times pr_2\sigma = \rho * \sigma$, if $\pi \neq \varnothing$, and $\rho * \pi * \sigma = \varnothing$ otherwise. It follows that identity (2) holds. It also follows that if $\varnothing \notin \Phi$, then $(\Phi, *)$ satisfies identity (6). If $\varnothing \in \Phi$, then $\varnothing$ is a zero element and axioms (7) and (8) hold. Thus, according to Lemmas 1 and 2, we obtain that all conditions of Theorem and Corollaries 1–3 are necessary.

Further, it is easy to see that for $U \neq \varnothing$, the Cartesian square of the semigroup $(\mathrm{Rel}(U), *)$ of relations contains the zero element $(\varnothing, \varnothing)$ and does not satisfy axiom (7). It follows that the classes $\mathrm{R}\{*\}$ and $\mathrm{R}\{*, \subseteq\}$ do not form quasi-varieties.

Step 3. The proof of the sufficiency of conditions of the Theorem is based on the result of [10]. Let us give some definitions and notations to formulate this result. For any formula $\varphi(z_0, z_1, r_1, \ldots, r_m)$ of the first-order predicate calculus having m binary predicate symbols $r_1, \ldots, r_m$ and two free individual variables z_0, z_1, we can associate an m-ary operation F_φ on $\mathrm{Rel}(U)$ defined in the following way:

$$F_\varphi(\rho_1, \ldots, \rho_m) = \{(u, v) \in U \times U : \varphi(u, v, \rho_1, \ldots, \rho_m)\},$$

where $\varphi(u, v, \rho_1, \ldots, \rho_m)$ means that the formula φ holds whenever z_0, z_1 are interpreted as u, v, and $r_1, \ldots, r_m$ are interpreted as relations $\rho_1, \ldots, \rho_m$ from $\mathrm{Rel}(U)$. Recall that an operation on relations is called primitive-positive if it can be defined by a first-order formula containing in its prenex normal form only existential quantifiers and conjunctions. Let us describ primitive-positive operations by using graphs [9].

Let $\mathbb{N}$ be the set of all natural numbers and $\mathbb{N}^0 = \mathbb{N} \cup \{0\}$. A *labeled graph* is a pair $G = (V(G),\ E(G))$, where $V(G)$ is a finite set, called a vertex set, and $E(G) \subseteq V(G) \times \mathbb{N} \times V(G)$ is a ternary relation. A triple $(u, k, v) \in E(G)$ is called an edge from u to v labeled by k, and it will be graphically represented by $u\cdot \overset{k}{\rightarrow} \cdot v$. An *input-output-pointed labeled graph* is a structure $G = (V(G), E(G), \mathrm{in}(G), \mathrm{out}(G))$, where $(V(G), E(G))$ is a labeled graph, $\mathrm{in}(G)$ and $\mathrm{out}(G)$ are two distinguished vertices (not necessarily different) called input and output vertices respectively. The input-output-pointed labeled graph G with $\mathrm{in}(G) = i$ and $\mathrm{out}(G) = o$ is also denoted by $G^{i,o}$. In what follows, we shall usually speak simply of graphs if it does not lead to confusion. The concept of graph isomorphism is defined in a natural way. All graphs will be considered up to isomorphism.

For given $u \in V(G)$, the number of edges of the form (u, k, v) [respectively, (v, k, u)] we denote by $deg^+(u)$ [respectively, $deg^-(u)$].

Given two input-output-pointed labeled graphs $G_1 = (V_1, E_1, \mathrm{in}_1, \mathrm{out}_1)$ and $G_2 = (V_2, E_2, \mathrm{in}_2, \mathrm{out}_2)$, a mapping $f\colon V_2 \to V_1$ is called a homomorphism from G_2 to G_1 if $f(\mathrm{in}_2) = \mathrm{in}_1$, $f(\mathrm{out}_2) = \mathrm{out}_1$, and $(f(u), k, f(v)) \in E_1$ whenever $(u, k, v) \in E_2$. We write $G_1 \prec G_2$ if there exists a homomorphism from G_2 to G_1.

Let $F = F_\varphi$ be a primitive positive operation determined by a formula φ. Then the input-output-pointed labeled graph $G = G_F = G_\varphi$ associated with F is defined as follows (see [5]). Let $\{0, 1, \dots, n\}$ be the set of all subscripts of individual variables of φ. Put $G = (V, E, \mathrm{in}, \mathrm{out})$, where $V = \{v_0, v_1, \dots, v_n\}$; $\mathrm{in} = v_0$, $\mathrm{out} = v_1$; $(i, k, j) \in E$ if and only if the atomic formula $r_k(z_i, z_j)$ occurs in φ.

Note that the graph $G_* = (V, E, \mathrm{in}, \mathrm{out})$ corresponding to the considered operation $*$ of the rectangular product can be described in the following way:

$$V = \{v_0, v_1, v_2, v_3\}, \quad E = \{(v_0, 1, v_3), (v_2, 2, v_1)\}, \ \mathrm{in} = v_0, \quad \mathrm{out} = v_1,$$

$$\mathrm{in} = v_0 \cdot \xrightarrow{(1)} \cdot v_3 \quad v_2 \cdot \xrightarrow{2} \cdot v_1 = \mathrm{out}.$$

Let $G = (V, E, \mathrm{in}, \mathrm{out})$ and $G_k = (V_k, E_k, \mathrm{in}_k, \mathrm{out}_k)$ $(k = 1, 2, \dots, n)$ be graphs with pairwise disjoint vertex sets. The composition $G(G_1, G_2, \dots, G_n)$ is the graph constructed as follows [5]: take G and substitute every edge $(u, k, v) \in E$ by the graph G_k identifying the input vertex in_k with u and the output vertex out_k with v.

For any semigroup term p define the graph $G(p) = (V(p), E(p), \mathrm{in}(p), \mathrm{out}(p))$ in the following inductive way:

1) if $p = x_k$, then $G(p)$ is the following graph: $\mathrm{in} \cdot \xrightarrow{k} \cdot \mathrm{out}$;

2) if $p = p_1 p_2$, then $G(p) = G_*(G(p_1), G(p_2))$.

According to the construction, for any term $p = x_{j_1} x_{j_2} \dots x_{j_{m-1}} x_{j_m}$ the graph $G(p)$ has the following form:

$$\mathrm{in} = v_0 \cdot \xrightarrow{j_1} \cdot \quad \cdot \xrightarrow{j_2} \cdot \quad \dots \quad \cdot \xrightarrow{j_{m-1}} \cdot \quad \cdot \xrightarrow{j_m} \cdot v_1 = \mathrm{out}.$$

Let G be a labeled graph, $u, v \in V(G) = \{v_0, v_1, \dots, v_n\}$, and Q be an input-output-pointed labeled graph. Without loss of generality, we can suppose that $V(Q) = \{w_0, w_1, \dots, w_m\}$, $\mathrm{in}(Q) = w_0 = u$, $\mathrm{out}(Q) = w_1 = v$, and $V(G) \cap V(Q) = \{u, v\}$. The labeled graph $(V(G) \cup V(Q), E(G) \cup E(Q))$ denote by $G[u, v, Q]$. Note that the edges set of $G[u, v, Q]$ can be represented as $\{v_0, v_1, \dots, v_n, v_{n+1}, \dots, v_{n+m-1}\}$, where $v_{n+1} = w_2, \dots, v_{n+m-1} = w_m$. Factually, the graph $G[u, v, Q]$ is obtained from the graph G by "gluing" the graph Q to the vertices u and v.

Define an *n-system* to be a pair $\omega = (\alpha, \beta)$, where $\alpha, \beta : \{1, \dots, n\} \to \mathbb{N}^0$ are mappings, $\alpha(k)$, $\beta(k) < 2 + (k-1)(m-2)$ for all $k = 1, \dots, n$, and m is the number of vertices of the graph that determines the considered operation on relations (for the operation $*$ we have $m = 4$ and $\alpha(k), \beta(k) < 2k$).

Given an n-system $\omega = (\alpha, \beta)$, construct by induction the sequence of graphs $G_0 \subseteq \dots \subseteq G_n = G_\omega$. Put G_0: $v_0 \cdot \xrightarrow{(1)} \cdot v_1$, and for $k = 1, \dots, n$ put: $G_k = G_{k-1}[v_{\alpha(k)}, v_{\beta(k)}, G(x_{2k} x_{2k+1})]$.

The following proposition presents the result of [11] formulated for the class $\mathrm{R}\{*, \subseteq\}$. This result gives an infinite basis of quasi-identities for the quasi-variety $\mathrm{Q}\{*, \subseteq\}$.

Proposition. *A partially ordered semigroup $(A, \cdot, \leqslant)$ belongs to the quasi-variety $\mathrm{Q}\{*, \subseteq\}$ if and only if it satisfies the quasi-identity*

$$(\bigwedge_{k=1}^{n} p_k \leqslant x_{2k} x_{2k+1}) \Rightarrow x_1 \leqslant p_0 \qquad (14)$$

for every n-system $\omega = (\alpha, \beta)$ and arbitrary terms $p_0, \dots, p_n$ such that $G_\omega^{v_0, v_1} \prec G(p_0)$ and $G_{k-1}^{v_{\alpha(k)}, v_{\beta(k)}} \prec G(p_k)$.

Step 4. We are ready to prove the sufficiency of the conditions of the Theorem. Let $\omega = (\alpha, \beta)$ be the n-system and $p_0, p_1, \dots, p_n$ be the terms such that $G_\omega^{v_0, v_1} \prec G(p_0)$ and $G_{k-1}^{v_{\alpha(k)}, v_{\beta(k)}} \prec G(p_k)$ for $k = 1, \dots n$. This system corresponds to the sequence graphs $G_0 \subseteq G_1 \subseteq \dots \subseteq G_n = G_\omega$, where $G_k = (V_k, E_k)$ for $k = 0, \dots, n$. According to the construction, for any $k \leqslant n$ we have that $V_k = \{v_0, v_1, \dots, v_{2k}, v_{2k+1}\}$ and

$$E_k = \{(v_0, 1, v_1)\} \cup \{(v_{\alpha(i)}, 2i, v_{2i+1}), (v_{2i}, 2i+1, v_{\beta(i)}) : i = 1, \dots, k\}.$$

Let us Proof by induction on k that $\alpha(k)$ is even, $\beta(k)$ is odd, and $deg^- v_{2i} = 0$, $deg^+ v_{2i} > 0$, $deg^- v_{2i+1} > 0$, $deg^+ v_{2i+1} = 0$ for any $i = 0, \dots, k$.

Let $k = 1$. Since $G_0^{v_{\alpha(1)}, v_{\beta(1)}} \prec G(p_1)$, we have that $p_1 = x_1$ or $p_1 = x_1^2$, and $\alpha(1) = 0$, $\beta(1) = 1$, $deg^- v_0 = 0$, $deg^+ v_0 = 2$, $deg^- v_1 = 2$, $deg^+ v_1 = 0$, $deg^- v_2 = 0$, $deg^+ v_2 = 1$, $deg^- v_3 = 1$, $deg^+ v_3 = 0$.

Suppose now that it holds for $k - 1$, and let us show that this is true for k. Since $G_{k-1}^{v_{\alpha(k)}, v_{\beta(k)}} \prec G(p_k)$, according to the definition of a graph homomorphism, we get that $deg^+ v_{\alpha(k)} > 0$, $deg^- v_{\beta(k)} > 0$. Then according to the induction assumption we get $\alpha(k)$ is even, $\beta(k)$ is odd, and $deg^- v_{2i} = 0$, $deg^+ v_{2i} > 0$, $deg^- v_{2i+1} > 0$, $deg^+ v_{2i+1} = 0$ for any $i = 0, \dots, k$.

Let $(A, \cdot, \leqslant)$ be a partially ordered semigroup satisfying identities (1) and (2). Suppose that the premise of quasi-identity (14) holds for some values of the variables $x_1 = a_1$, $x_2 = a_2$, $x_3 = a_3$, $\dots, x_{2n} = a_{2n}$, $x_{2n+1} = a_{2n+1}$, i.e., $p_k(\vec{a}) \leqslant a_{2k} a_{2k+1}$ for all $k = 1, \dots, n$, where $\vec{a} = (a_1, a_2, \dots, a_{2n+1})$. Let $p_0 = x_{j_1} x_{j_2} \dots x_{j_{m-1}} x_{j_m}$. Note that $G_k^{v_0, v_1} \prec G(p_0)$ if and only if $\{x_{j_1}, x_{j_2}, \dots, x_{j_{m-1}}, x_{j_m}\} \subseteq \{x_1, x_2, \dots, x_{2k}, x_{2k+1}\}$, $x_{j_m} = x_{2j+1}$ for some $j \leqslant k$ such that $\beta(j) = 1$, and $x_{j_1} = x_1$ or $x_{j_1} = x_{2i}$ for some $i \leqslant k$ such that $\alpha(i) = 0$. It follows that the equality $x_{j_1} = x_{j_m}$ is possible only if $x_{j_1} = x_{j_m} = x_1$, otherwise we can assume that all variables of p_0 are different.

Let $max(p_0)$ be the greatest k such that at least one of the variables x_{2k} or x_{2k+1} is included in the term p_0. Let us prove by induction on $max(p_0)$ that $a_1 \leqslant p_0(\vec{a})$. If $max(p_0) = 0$ then $p = x_1$ or $p_0 = x_1^2$. Thus, using identity (1) we obtain $a_1 \leqslant p_0(\vec{a})$. Suppose now that $a_1 \leqslant p_0(\vec{a})$ holds for $max(p_0) = k - 1$, and let us show that this is true for $max(p_0) = k$.

If both variables x_{2k}, x_{2k+1} are included in p_0, then the following cases are possible:

1) $p_0 = x_{2k} x_{2k+1} x_{j_3} \dots x_{j_{m-1}} x_{j_m}$, then using the induction assumption we get

$$a_0 \leqslant p_k(\vec{a}) a_{j_3} \dots a_{j_{m-1}} a_{j_m} \leqslant a_{2k} a_{2k+1} a_{j_3} \dots a_{j_{m-1}} a_{j_m} = p_0(\vec{a});$$

2) $p_0 = x_{j_1} x_{2k} x_{2k+1} x_{j_4} \dots x_{j_{m-1}} x_{j_m}$, then using the induction assumption we get

$$a_0 \leqslant a_{j_1} p_k(\vec{a}) a_{j_4} \dots a_{j_{m-1}} a_{j_m} \leqslant a_{j_1} a_{2k} a_{2k+1} a_{j_4} \dots a_{j_{m-1}} a_{j_m} = p_0(\vec{a});$$

3) $p_0 = x_{j_1} x_{j_2} \dots x_{j_{m-3}} x_{2k} x_{2k+1}$, then using the induction assumption we get

$$a_0 \leqslant a_{j_1} a_{j_2} \dots a_{j_{m-3}} p_k(\vec{a}) \leqslant a_{j_1} a_{j_2} \dots a_{j_{m-3}} a_{2k} a_{2k+1} = p_0(\vec{a});$$

4) $p_0 = x_{2k} x_{j_2} \dots x_{j_{m-1}} x_{2k+1}$, then $\alpha(k) = 0$ and $\beta(k) = 1$. It follows that

$$G_k^{v_0, v_1} \prec G(p_k x_{j_2} \dots x_{j_{m-1}} p_k),$$

and by the induction assumption we get

$$a_0 \leqslant p_k(\vec{a}) a_{j_2} \dots a_{j_{m-1}} p_k(\vec{a}) \leqslant a_{2k} a_{2k+1} a_{j_2} \dots a_{j_{m-1}} a_{2k} a_{2k+1} \overset{(2)}{\leqslant}$$
$$\leqslant a_{2k} a_{j_2} \dots a_{j_{m-1}} a_{2k+1} = p_0(\vec{a}).$$

If only one of the variables x_{2k} or x_{2k+1} is included in p_0, then the following cases are possible:

5) $p_0 = x_{2k} x_{j_2} \dots x_{j_{m-1}} x_{j_m}$, then using the induction assumption we get

$$a_0 \leqslant p_k(\vec{a}) a_{j_3} \dots a_{j_{m-1}} a_{j_m} \leqslant a_{2k} a_{2k+1} a_{j_3} \dots a_{j_{m-1}} a_{j_m} \overset{(2)}{\leqslant} a_{2k} a_{j_3} \dots a_{j_{m-1}} a_{j_m} = p_0(\vec{a});$$

6) $p_0 = x_{j_1} x_{2k} x_{j_3} \dots x_{j_{m-1}} x_{j_m}$, then using the induction assumption we get

$$a_0 \leqslant a_{j_1} p_k(\vec{a}) a_{j_3} \dots a_{j_{m-1}} a_{j_m} \leqslant a_{j_1} a_{2k} a_{2k+1} a_{j_3} \dots a_{j_{m-1}} a_{j_m} \overset{(2)}{\leqslant} a_{j_1} a_{2k} a_{j_3} \dots a_{j_{m-1}} a_{j_m} = p_0(\vec{a});$$

7) $p_0 = x_{j_1}x_{2k+1}x_{j_3}\ldots x_{j_{m-1}}x_{j_m}$, then using the induction assumption we get

$$a_0 \leqslant a_{j_1}p_k(\vec{a})a_{j_4}\ldots a_{j_{m-1}}a_{j_m} \leqslant a_{j_1}a_{2k}a_{2k+1}a_{j_4}\ldots a_{j_{m-1}}a_{j_m} \overset{(2)}{\leqslant} \\ \leqslant a_{j_1}a_{2k+1}a_{j_4}\ldots a_{j_{m-1}}a_{j_m} = p_0(\vec{a});$$

8) $p_0 = x_{j_1}x_{j_2}\ldots x_{j_{m-1}}x_{2k+1}$, then using the induction assumption we get

$$a_0 \leqslant a_{j_1}a_{j_2}\ldots a_{j_{m-1}}p_k(\vec{a}) \leqslant a_{j_1}a_{j_2}\ldots a_{j_{m-1}}a_{2k}a_{2k+1} \overset{(2)}{\leqslant} a_{j_1}a_{j_2}\ldots a_{j_{m-1}}a_{2k+1} = p_0(\vec{a}).$$

Thus, we have proved that the partially ordered semigroup $(A,\cdot,\leqslant)$ satisfies quasi-identities (14). Therefore, according to the Proposition we have $(A,\cdot,\leqslant) \in \mathrm{Q}\{*,\subseteq\}$. This completes the proof of the Theorem.

Step 4. Let us prove the sufficiency of the conditions of Corollary 1. Suppose that a semigroup $(A,\cdot)$ satisfies identities (3)–(5) and $A^2 = \{a^2 : \ a \in A\}$. We define the relation $\leqslant$ on the set A by setting

$$\leqslant = \{(x,y) \in A \times A^2 : \ x^2 = yxy\} \cup \{(x,x) \in A \times A : \ x \in A\}.$$

Let us show that $(A,\cdot,\leqslant)$ is the partially ordered semigroup satisfying identities (1) and (2). The reflexivity of the relation $\leqslant$ follows directly from the definition.

To prove the transitivity assume that $x \leqslant y$ and $y \leqslant z$. Without loss of generality, we can suppose that $x \neq y$ and $y \neq z$. Then $x^2 = yxy$, $y^2 = zyz$ and $y^2 = y$, $z^2 = z$, hence $x^2 = yxy = zyzxzyz \overset{(11)}{=} zzyxyzz = zyxyz = zx^2z \overset{(12)}{=} zxz$, i.e., $x \leqslant z$. Thus, $\leqslant$ is transitive.

Assume that $x \leqslant y$, $y \leqslant x$ and $x \neq y$. Then $x^2 = yxy$, $y^2 = xyx$ and $x^2 = x$, $y^2 = y$, hence $x = x^2 = yxy = y^2xy^2 = xyxxxyx \overset{(11)}{=} xyx = y^2 = y$. This contradicts the assumption $x \neq y$. Thus, $\leqslant$ is a partially order relation.

Let us show that the relation $\leqslant$ is compatible with multiplication. Suppose that $x \leqslant y$ and $x \neq y$. Then $x^2 = yxy$ and $y^2 = y$, hence $(xz)^2 \overset{(3)}{=} xz \overset{(13)}{=} x^2z = yxyz \overset{(12)}{=} yxyz^3 \overset{(11)}{=} yzxzyz$ and $(yz)^2 \overset{(3)}{=} yz$. Thus, $xz \leqslant yz$. Further, $(zx)^2 \overset{(3)}{=} zx \overset{(12)}{=} zx^2 = zyxy \overset{(13)}{=} z^3yxy \overset{(11)}{=} zyzxzy$ and $(zy)^2 \overset{(3)}{=} zy$. Thus, $zx \leqslant zy$.

Since $x^2 \overset{(12)}{=} x^2xx^2$ and $(x^2)^2 \overset{(3)}{=} x^2$, we have $x \leqslant x^2$. Since $(xyz)^2 \overset{(3)}{=} xyz \overset{(12)}{=} xyz^3 \overset{(13)}{=} \\ = x^3yz^3 \overset{(11)}{=} xzxyzxz$ and $(xz)^2 \overset{(3)}{=} xz$, we have $xyz \leqslant xz$. Therefore, $(A,\cdot,\leqslant)$ satisfies identities (1) and (2), hence $(A,\cdot,\leqslant) \in \mathrm{Q}\{*,\subseteq\}$ and $(A,\cdot) \in \mathrm{Q}\{*\}$. This completes the proof of Corollary 1.

Step 5. Let us prove the sufficiency of the conditions of Corollary 2 and 3.

Lemma 3. *Let $\{U_j : j \in J\}$ be a family of pairwise non-intersecting sets and $U = \\ = \bigcup\{U_j : j \in J\}$. If a partially ordered semigroup $(A,\cdot,\leqslant)$ is a subdirect product of a family $\{(\Phi_j,*,\subseteq) : j \in J\}$ of partially ordered semigroups of relations on U_j, and satisfies identity (6), then $(A,\cdot,\leqslant)$ isomorphically embedded in $(\mathrm{Rel}(U),*,\subseteq)$.*

Proof. Let $\varphi_j : A \longrightarrow \Phi_j$ be the corresponding surjective homomorphisms from A on the components of the direct product $\prod\{\Phi_j : j \in J\}$. According to the properties of homomorphic images, we see that all components $(\Phi_j,*,\subseteq)$ satisfy identity (6). Hence, for all $j \in J$ we have $\varnothing \notin \Phi_j$ or $\Phi_j = \{\varnothing\}$. It follows that $(A,\cdot,\leqslant)$ is subdirect product of the family $\{(\Phi_j,*,\subseteq) : j \in J_0\}$, where $J_0 = \{j \in J : \varnothing \notin \Phi_j\}$.

For given $a \in A$, put $\rho_j^a = \varphi_j(a)$. We define a mapping $\varphi : A \longrightarrow \mathrm{Rel}(U)$ in the following way. Put $\varphi(a) = \bigcup\{pr_1\rho_j^a : j \in J_0\} \times \bigcup\{pr_2\rho_j^a : j \in J_0\}$, if $a^2 = a$, and $\varphi(a) = \bigcup\{\rho_j^a : j \in J_0\} \cup \bigcup\{\varphi(b) : b^2 = b \leqslant a\}$ otherwise. Let us show that φ is an isomorphic embedding $(A,\cdot,\leqslant)$ in $(\mathrm{Rel}(U),*,\subseteq)$.

Note that $\varphi(a) \cap U_j \times U_j = \rho_j^a$ for all $a \in A$. It follows that $\varphi(a) \subseteq \varphi(a)$ if and only if $a \leqslant b$. Forver, since $(ab)^2 = ab$, we have

$$\varphi(ab) = \bigcup\{pr_1\rho_j^{ab} : j \in J_0\} \times \bigcup\{pr_2\rho_j^{ab} : j \in J_0\} =$$

$$= \bigcup\{pr_1(pr_1\rho_j^a \times pr_2\rho_j^b) : j \in J_0\} \times \bigcup\{pr_2(pr_1\rho_j^a \times pr_2\rho_j^b) : j \in J_0\} =$$

$$= (\bigcup\{pr_1\rho_j^a : j \in J_0\} \times \bigcup\{pr_2\rho_j^b : j \in J_0\}) = pr_1\varphi(a) \times pr_2\varphi(b) = \varphi(a) * \varphi(b).$$

□

Lemma 4. *Suppose that* $(A, \cdot, \leqslant)$ *satisfies identities* (1) *and* (6). *Then* $(A, \cdot, \leqslant)$ *belongs to* $\mathrm{R}\{*, \subseteq\}$.

Proof. If $(A, \cdot, \leqslant)$ satisfies identities (1) and (6), then according to the Theorem we have $(A, \cdot, \leqslant) \in \mathrm{Q}\{*, \subseteq\}$. In respect that the class $\mathrm{R}\{*, \subseteq\}$ is axiomatizable [34], we obtain that $(A, \cdot, \leqslant)$ is a subdirect product of a family of partially ordered semigroups from $\mathrm{R}\{*, \subseteq\}$. Hence, according to Lemma 3, we obtain that $(A, \cdot, \leqslant)$ belongs to $\mathrm{R}\{*, \subseteq\}$. □

Lemma 5. *Suppose that* $(A, \cdot)$ *satisfies identity* (6). *Then* $(A, \cdot)$ *belongs to* $\mathrm{R}\{*\}$.

Proof. If $(A, \cdot)$ satisfies identity (6), then according to Lemma 1 it also satisfies identities (3)–(5). Let $\leqslant$ be the partial order relation constructed in the proof of Corollary 1. Then by Lemma 4, we have $(A, \cdot, \leqslant) \in \mathrm{R}\{*, \subseteq\}$. Therefore, $(A, \cdot) \in \mathrm{R}\{*\}$. □

Lemma 6. *Suppose that* $(A, \cdot)$ *contains the zero element* o *and satisfies axiom* (7). *Then* $(A, \cdot)$ *satisfies identities* (6) *or* $ab \neq o$ *for all* $a, b \neq o$.

Proof. If there exist elements $a \neq o$ and $b \neq o$ such that $ab = o$, then for all $x, y \neq o$ we have $xay \overset{(7)}{=} xy$, $xby \overset{(7)}{=} xy$, and $xaby = o$, hence $xy \overset{(7)}{=} xyxy = xayxby \overset{(7)}{=} xaby = xoy = o$. It follows that $xyz = xz$ for all $x, y, z \in A$, i.e., $(A, \cdot)$ satisfies identities (6). □

Suppose that $(A, \cdot, \leqslant)$ contains the zero element o and satisfies identity (1) and axioms (7) and (8). Put $B = A \backslash \{o\}$. According to Lemmas 4 and 6, we can suppose that $xy \in B$ for all $x, y \in B$, and $(B, \cdot, \leqslant)$ satisfies identities (1) and (6), hence $(B, \cdot, \leqslant)$ belongs to $\mathrm{R}\{*, \subseteq\}$. It means that there exists an isomorphism F from the partially ordered semigroup $(B, \cdot, \leqslant)$ to some partially ordered semigroup of relations $(\Phi, *, \subseteq)$ and $\varnothing \notin \Phi$. Putting $F(o) = \varnothing$, we get the isomorphism from $(A, \cdot, \leqslant)$ to $(\Phi \cup \{\varnothing\}, *, \subseteq)$. Therefore, $(A, \cdot, \leqslant)$ belongs to $\mathrm{R}\{*, \subseteq\}$. This completes the proof of Corollary 2.

Suppose that $(A, \cdot)$ contains the zero element o and satisfies axiom (7), $B = A \backslash \{o\}$, and let $\leqslant$ be the partial order relation on B constructed in the proof of Corollary 1. Extend the relation $\leqslant$ on A by putting $o \leqslant a$ for all $a \in A$. Then $(A, \cdot, \leqslant)$ satisfies the condition 3 of the Theorem, hence $(A, \cdot, \leqslant) \in \mathrm{R}\{*, \subseteq\}$. Therefore, $(A, \cdot)$ belongs to $\mathrm{R}\{*\}$. This completes the proof of Corollary 3.

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