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Article

On recovering non-local perturbation of non-self-adjoint Sturm – Liouville operator

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Abstract. Recently, there appeared a significant interest in inverse spectral problems for non-local operators arising in numerous applications. In the present work, we consider the operator with frozen argument $ly = -y''(x) + p(x)y(x) + q(x)y(a)$, which is a non-local perturbation of the non-self-adjoint Sturm – Liouville operator. We study the inverse problem of recovering the potential $q \in L_2(0, \pi)$ by the spectrum when the coefficient $p \in L_2(0, \pi)$ is known. While the previous works were focused only on the case $p = 0$, here we investigate the more difficult non-self-adjoint case, which requires consideration of eigenvalues multiplicities. We develop an approach based on the relation between the characteristic function and the coefficients $\{\xi_n\}_{n\geqslant 1}$ of the potential q by a certain basis. We obtain necessary and sufficient conditions on the spectrum being asymptotic formulae of a special form. They yield that a part of the spectrum does not depend on q, i.e. it is uninformative. For the unique solvability of the inverse problem, one should supplement the spectrum with a part of the coefficients ξ_n, being the minimal additional data. For the inverse problem by the spectrum and the additional data, we obtain a uniqueness theorem and an algorithm.
Keywords: inverse spectral problems, frozen argument, Sturm – Liouville operators, non-local operators, necessary and sufficient conditions
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Восстановление нелокального возмущения несамосопряженного оператора Штурма – Лиувилля

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Аннотация. В последнее время возник значительный интерес к обратным спектральным задачам для нелокальных операторов, возникающих во многих приложениях. В настоящей работе рассматривается оператор с замороженным аргументом $ly = -y''(x) + p(x)y(x) + q(x)y(a)$, который является нелокальным возмущением несамосопряженного оператора Штурма – Лиувилля. Исследуется обратная задача восстановления потенциала $q \in L_2(0, \pi)$ по спектру при известном коэффициенте $p \in L_2(0, \pi)$. В то время как предыдущие работы были сосредоточены только на случае $p = 0$, здесь исследуется более сложный несамосопряженный случай, требующий учета кратностей собственных значений. Мы развиваем подход, основанный на связи между характеристической функцией и коэффициентами $\{\xi_n\}_{n\geqslant 1}$ потенциала q по некоторому базису. Получены необходимые и достаточные условия для спектра, которые являются асимптотическими формулами особого вида. Из них следует, что часть спектра не зависит от q, т. е. является неинформативной. Для однозначной разрешимости обратной задачи кроме спектра необходимо задать часть коэффициентов ξ_n, которые являются минимальными дополнительными данными. Для обратной задачи по спектру и дополнительным данным получены теорема единственности и алгоритм.

Ключевые слова: обратные спектральные задачи, замороженный аргумент, операторы Штурма – Лиувилля, нелокальные операторы, необходимые и достаточные условия

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Introduction

Inverse spectral problems consist of recovering operators from their spectral characteristics. The classical results in this field were obtained for the differential operators [1–5], which are local. Recently, in connection with numerous applications, there appeared a considerable interest in inverse problems for non-local operators [6–12], and, in particular, for the operators with a frozen argument [13–25]. Their studying is complicated by the fact that non-local operators require the development of non-standard methods.

In this paper, we study the recovery of a complex-valued potential $q \in L_2(0, \pi)$ by the spectrum $\{\lambda_n\}_{n\geqslant 1}$ of the boundary value problem

$$ly := -y''(x) + p(x)y(x) + y(a)q(x) = \lambda y(x), \quad x \in (0, \pi), \tag{1}$$

$$y^{(\alpha)}(0) = y^{(\beta)}(\pi) = 0, \quad \alpha, \beta \in \{0, 1\}, \tag{2}$$

where $a \in [0, \pi]$ and $p \in L_2(0, \pi)$ is complex-valued. Operators ly are usually called Sturm – Liouville operators with *frozen argument*. They have a close relation to the operators with integral boundary conditions [8, 12, 26–28], which arise in studying diffusion and heating processes and in the theory of elasticity, see [29–32]. In connection with this, in [33], some spectral properties of the operator ly were established in the case of periodic boundary conditions. However, the mentioned work does not address an inverse spectral problem.

The previous studies of inverse spectral problems for Sturm – Liouville operators with frozen argument [13–25] were focused only on the case $p = 0$. In this particular case, a comprehensive study of recovering q by the spectrum required a series of works [15–18, 20, 22, 25]. The most general approach to the operator with frozen argument was developed in [22], which allowed us to obtain necessary and sufficient conditions on the spectrum and, afterward, a uniform stability of the inverse problem [25].

In [23], there was suggested another approach to the operator with frozen argument within the framework of perturbation theory. According to it, operator ly could be treated as a one-dimensional perturbation of the Sturm – Liouville operator $Ay := -y'' + p(x)y$. However, for studying the spectral properties of ly, this approach needs the selfadjointness of the operator A. For this reason, it is inapplicable to the case of complex-valued p considered here, since the unperturbed operator A is non-self-adjoint.

Here, we develop an approach to the general situation of arbitrary $p \in L_2(0, \pi)$, relying on some ideas of [22]. We significantly extend the mentioned ideas to take into account the non-self-adjointness of the unperturbed operator, which requires consideration of eigenvalues multiplicities.

The main results of the paper consist in necessary and sufficient conditions on the spectrum, a uniqueness theorem, and an algorithm. As in the previous work [22], the necessary and sufficient conditions are asymptotic formulae of a special form. They give the so-called degeneration condition that some part of the eigenvalues is uninformative, i.e. it does not depend on q. However, compared to [22], the non-self-adjointness of the unperturbed operator leads to a new effect that the degeneration condition includes a restriction on the minimal possible multiplicity of each uninformative eigenvalue.

The paper is organized as follows. In Section 1, we introduce necessary objects and provide auxiliary statements. As well we obtain a characteristic function of boundary value problems (1), (2), and a main equation of the inverse problem. In Section 2, by necessity, we establish the conditions on the spectrum. The main results and their proofs are given in Section 3.

1. Preliminaries. The main equation of the inverse problem

First, we consider the unperturbed boundary value problem for the classical Sturm – Liouville equation

$$-y''(x) + p(x)y(x) = \lambda y(x), \quad x \in (0, \pi), \tag{3}$$

with boundary conditions (2). Denote by $S_a(x, \lambda)$ and $C_a(x, \lambda)$ the solutions of equation (3) under the initial conditions

$$C_a(a, \lambda) = 1, \; C'_a(a, \lambda) = 0, \quad S_a(a, \lambda) = 0, \; S'_a(a, \lambda) = 1;$$

here and below the prime symbol stands for the derivative with respect to the *first* argument. Let us agree that $\int_a^b f(t)\,dt$ is understood as $-\int_b^a f(t)\,dt$ when $b < a$. For $x \in [0, \pi]$, using transformation operators (see, e.g. [4]), we obtain the following representations:

$$\left.\begin{aligned}
S_a(x, \lambda) &= \frac{\sin\rho(x-a)}{\rho} - \frac{\cos\rho(x-a)}{\rho^2}\omega_a(x) + \int_a^x K_{a,1}(x,t)\frac{\cos\rho(t-a)}{\rho^2}\,dt,\\
S'_a(x, \lambda) &= \cos\rho(x-a) + \frac{\sin\rho(x-a)}{\rho}\omega_a(x) + \int_a^x K_{a,2}(x,t)\frac{\sin\rho(t-a)}{\rho}\,dt,\\
C_a(x, \lambda) &= \cos\rho(x-a) + \frac{\sin\rho(x-a)}{\rho}\omega_a(x) + \int_a^x K_{a,3}(x,t)\frac{\sin\rho(t-a)}{\rho}\,dt,\\
C'_a(x, \lambda) &= -\rho\sin\rho(x-a) + \cos\rho(x-a)\omega_a(x) + \int_a^x K_{a,4}(x,t)\cos\rho(t-a)\,dt,
\end{aligned}\right\} \tag{4}$$

where $\rho^2 = \lambda$ and $\omega_a(x) = \frac{1}{2}\int_a^x p(t)\,dt$. In (4), for each fixed x, there determined $K_{a,j}(x, \cdot) \in$ $\in L_2(0, \pi)$, and $K_{a,j} \in L_2[0, \pi]^2$, $j = \overline{1, 4}$.

Consider the entire function $\Delta_0(\lambda) = C_a^{(\alpha)}(0, \lambda)S_a^{(\beta)}(\pi, \lambda) - S_a^{(\alpha)}(0, \lambda)C_a^{(\beta)}(\pi, \lambda)$, wherein $y^{(0)} := y$ and $y^{(1)} := y'$. It is easy to see that Δ_0 is a characteristic function of unperturbed boundary value problem (2), (3), i.e. a number μ_n is its eigenvalue if and only if $\Delta_0(\mu_n) = 0$. By $\{\mu_n\}_{n \geqslant 1}$ we denote the spectrum of (2), (3), being the sequence of the eigenvalues taken with the account of algebraic multiplicities. The following asymptotics are known (see [4]):

$$\mu_n = \theta_n^2, \quad \theta_n = n - \frac{\alpha+\beta}{2} + \frac{\omega}{\pi n} + \frac{\kappa_n}{n}, \quad n \geqslant 1, \quad \omega := \frac{1}{2}\int_0^\pi p(t)\,dt, \quad \{\kappa_n\}_{n\geqslant 1} \in \ell_2. \tag{5}$$

By m_n we denote multiplicity of the eigenvalue μ_n. By asymptotics (5), for a sufficiently large n, we have $m_n = 1$. Without loss of generality, we assume that equal values in the spectrum follow each other. Then, we have

$$\mu_n = \mu_{n+1} = \ldots = \mu_{n+m_n-1}, \quad n \in \mathcal{S}, \quad \mathcal{S} := \{n \geqslant 2\colon \mu_n \neq \mu_{n-1}\} \cup \{1\}.$$

The index $n \in \mathcal{S}$ corresponds to the unique elements in $\{\mu_n\}_{n\geqslant 1}$, while for $n \in \mathcal{S}$ and $\nu \in \overline{0, m_n - 1}$, the index $k = n + \nu$ runs through $\mathbb{N}$.

Now, we are ready to study boundary value problems (1) and (2). Introduce the solutions $S(x,\lambda)$ and $C(x,\lambda)$ of equation (1) under the initial conditions

$$S(a,\lambda) = 0, \quad S'(a,\lambda) = 1, \quad C(a,\lambda) = 1, \quad C'(a,\lambda) = 0.$$

Any other solution of (1) is a linear combination of $S(x,\lambda)$ and $C(x,\lambda)$. It is easy to see that

$$S(x,\lambda) = S_a(x,\lambda), \quad C(x,\lambda) = C_a(x,\lambda) + \int_a^x W(x,t,\lambda)q(t)\,dt, \tag{6}$$

where $W(x,t,\lambda) := C_a(t,\lambda)S_a(x,\lambda) - C_a(x,\lambda)S_a(t,\lambda)$. We introduce the entire function

$$\Delta(\lambda) = C^{(\alpha)}(0,\lambda)S^{(\beta)}(\pi,\lambda) - S^{(\alpha)}(0,\lambda)C^{(\beta)}(\pi,\lambda). \tag{7}$$

Then, $\Delta(\lambda)$ is a characteristic function of boundary value problem (1), (2), while $\{\lambda_n\}_{n\geqslant 1}$ is a sequence of its zeroes taken with the account of multiplicities.

Substituting (6) into (7), we obtain

$$\Delta(\lambda) = \Delta_0(\lambda) - S_a^{(\beta)}(\pi,\lambda)\int_0^a W^{(\alpha)}(0,t,\lambda)\,q(t)\,dt - S_a^{(\alpha)}(0,\lambda)\int_a^\pi W^{(\beta)}(\pi,t,\lambda)\,q(t)\,dt. \tag{8}$$

Following the approach in [22], we should substitute into (8) the values $\lambda = \mu_n$, being the zeroes of the main part $\Delta_0(\lambda)$. In the paper [22] corresponding to the case $p = 0$, we had $\mu_n = \left(n - \frac{\alpha+\beta}{2}\right)^2$, being simple eigenvalues. Here, we have to take into account that μ_n may be multiple. For each $n \in \mathcal{S}$, we differentiate both parts of formula (8) $\nu = \overline{0, m_n - 1}$ times and put $\lambda = \mu_n$. Since

$$\frac{\partial^\nu}{\partial\lambda^\nu}\left[C_a^{(\alpha)}(0,\lambda)S_a^{(\beta)}(\pi,\lambda)\right] = \frac{\partial^\nu}{\partial\lambda^\nu}[S_a^{(\alpha)}(0,\lambda)C_a^{(\beta)}(\pi,\lambda)], \quad n \in \mathcal{S},\ \nu = \overline{0, m_n - 1}, \tag{9}$$

in $\lambda = \mu_n$, we obtain

$$\Delta^{(\nu)}(\lambda) = \left[S_a^{(\beta)}(\pi,\lambda)\int_0^\pi g(t,\lambda)q(t)\,dt\right]^{(\nu)}, \quad \lambda = \mu_n,\ n \in \mathcal{S},\ \nu = \overline{0, m_n - 1}, \tag{10}$$

where $g(t,\lambda) := -W^{(\alpha)}(0,t,\lambda)$. One can see that $g(t,\lambda) = S_0(x,\lambda)$ if $\alpha = 0$, and $g(t,\lambda) = -C_0(x,\lambda)$ if $\alpha = 1$. Moreover, $g^{(\beta)}(\pi,\lambda) = \Delta_0(\lambda)$.

For $n \in \mathcal{S}$ and $\nu = \overline{0, m_n - 1}$, we introduce

$$a_{n+\nu} = \frac{n^{1-\beta}}{\nu!}\left.\frac{\partial^\nu S_a^{(\beta)}(\pi,\lambda)}{\partial\lambda^\nu}\right|_{\lambda=\mu_n}, \quad g_{n+\nu}(t) = \frac{n^{1-\alpha}}{(m_n-\nu-1)!}\left.\frac{\partial^{m_n-\nu-1}g(t,\lambda)}{\partial\lambda^{m_n-\nu-1}}\right|_{\lambda=\mu_n}.$$

Note that a_k and g_k are the objects constructed by unperturbed boundary value problems (2), (3), and they are known. Formulae (4) and (5) yield that $a_k = O(1)$ and $\{g_k(t)\}_{k\geqslant 1}$ is an almost normalized system. At the same time, $\{g_k(t)\}_{k\geqslant 1}$ is constructed from eigen- and associated functions of the operator $Ay = -y'' + p(x)y$ considered under strongly regular conditions (2). For the mentioned reasons, the following proposition holds (see, e.g., [34, 35]).

Proposition 1. *The functional sequence* $\{g_n(t)\}_{n\geqslant 1}$ *is a Riesz basis in* $L_2(0,\pi)$.

Applying the general Leibniz rule in (10), we get

$$n^{2-\alpha-\beta}\frac{\Delta^{(\nu)}(\mu_n)}{\nu!}=\sum_{\eta=0}^{\nu}a_{n+\nu-\eta}\xi_{n+m_n-1-\eta},\quad n\in\mathcal{S},\quad \nu=\overline{0,m_n-1},\tag{11}$$

where $\xi_k := \int_0^\pi g_k(t)q(t)\,dt$. Further, we consider (11) as an equation with respect to the coefficients $\{\xi_n\}_{n\geqslant 1}$. From Proposition 1 it follows that $\{\xi_n\}_{n\geqslant 1}\in\ell_2$ and its knowledge allows one to recover q uniquely. Formula (11) is called the *main equation* of the inverse problem.

2. Necessary conditions

In this section, we obtain necessary conditions on the spectrum $\{\lambda_n\}_{n\geqslant 1}$, which consists of asymptotic formulae (21). In the next section, we prove that they are sufficient conditions as well.

Remind that $\lambda_n=\rho_n^2$ and $\mu_n=\theta_n^2$. Without loss of generality, we assume that $\arg\rho_n$, $\arg\theta_n\in[-\frac{\pi}{2},\frac{\pi}{2})$. We start by obtaining the weakest asymptotics for ρ_n.

Lemma 1. *The following asymptotics holds:* $\rho_n=\theta_n+o(1)$, $n\geqslant 1$.

Proof. For definiteness, we provide computations in the case $\alpha=1$ and $\beta=0$ (the other cases are proceeded analogously). By asymptotics (5), it is sufficient to prove that $\rho_n=n-\frac{1}{2}+o(1)$. For $\lambda=\rho^2$, we consider $\Delta(\lambda)$ as an entire function of ρ. Then, $\{\rho_n\}_{n\geqslant 1}\cup\{-\rho_n\}_{n\geqslant 1}$ is the set of all its zeroes.

Denote $\tau=|\mathrm{Im}\,\rho|$. Using the corresponding formulae in (4), for $\rho\to\infty$, we obtain

$$W'(0,t,\lambda)=-C_0(t,\lambda)=O\left(e^{\tau t}\right),\quad W(\pi,t,\lambda)=-S_\pi(t,\lambda)=O\left(\frac{e^{\tau(\pi-t)}}{\rho}\right),$$

$$S_a(\pi,\lambda)=O\left(\frac{e^{\tau(\pi-a)}}{\rho}\right),\quad S_a'(0,\lambda)=O\left(e^{\tau a}\right),\quad \Delta_0(\lambda)=-C_0(\pi,\lambda)=-\cos\rho\pi+O\left(\frac{e^{\tau\pi}}{\rho}\right).$$

Substituting these relations into (8), we get

$$\Delta(\lambda)=\Delta_0(\lambda)+O\left(\frac{e^{\tau\pi}}{\rho}\right)=\cos\rho\pi+O\left(\frac{e^{\tau\pi}}{\rho}\right).\tag{12}$$

For any $\delta\in(0,\frac{1}{4})$, we have the following estimate with $C_\delta>0$ (see [4]):

$$|\cos\rho\pi|>C_\delta\, e^{\tau\pi},\quad \rho\in G_\delta:=\left\{z\in\mathbb{C}\colon\left|z-n+\frac{1}{2}\right|\geqslant\delta,\ n\in\mathbb{Z}\right\}.\tag{13}$$

Then, by (12), there exists $N_\delta\in\mathbb{N}$ such that $|\cos\rho\pi|\geqslant|\cos\rho\pi-\Delta(\rho^2)|$ as soon as $\rho\in G_\delta$ and $|\rho|\geqslant N_\delta$. Applying Rouche's theorem, we arrive at that in the circle $|\rho|<N_\delta$, the functions $\cos\rho\pi$ and $\Delta(\rho^2)$ have the same number of zeroes. Analogously, in each circle $|\rho-n+\frac{1}{2}|<\delta$, where $n\in\mathbb{Z}$ is such that $|n-\frac{1}{2}|>N_\delta$, there is exactly one zero of $\Delta(\rho^2)$. Taking into account that $\Delta(\rho^2)$ and $\cos\rho\pi$ are even functions of ρ, and that δ can be arbitrarily small, we arrive at the needed asymptotics. □

Further, we clarify the obtained in Lemma 1 necessary conditions on the spectrum.

Theorem 1. *I. Let* $K\in\mathbb{N}$ *be such that for* $n\geqslant K$, *all* $m_n=1$. *Then, the following asymptotics holds:*

$$\lambda_n=\mu_n+a_n\varkappa_n,\ n\geqslant K,\quad \{\varkappa_n\}_{n\geqslant K}\in\ell_2.\tag{14}$$

II. For $n\in\mathcal{S}$, *denote* $k_n=\min(m_n,r_n)$, *where* r_n *is a multiplicity of* $\lambda=\mu_n$ *as a zero of the entire function* $S_a^{(\beta)}(\pi,\lambda)$. *Then, there exists such numeration of* $\{\lambda_n\}_{n\geqslant 1}$ *that*

$$\lambda_n=\lambda_{n+1}=\ldots=\lambda_{n+k_n-1}=\mu_n,\quad n\in\mathcal{S}.\tag{15}$$

Proof. I. By Lemma 1, we have $\rho_n = \theta_n + \eta_n$, where $\eta_n = o(1)$. To prove (14), we substitute $\lambda = \rho_n^2$ into (8) and, using the Taylor series, obtain asymptotics for η_n, where $n \geqslant K$.

For definiteness, consider the case $\alpha = 1$ and $\beta = 0$ (the other cases are proceeded analogously). Since $\Delta_0(\lambda) = -C_0(\pi, \lambda)$, substituting $\lambda = \rho_n^2$ into the corresponding formula in (4), applying trigonometric formulae along with asymptotics (5) and $\eta_n = o(1)$, we get

$$\Delta_0(\lambda_n) = -\cos\theta_n\pi - \frac{\omega}{\theta_n}\sin\theta_n\pi - \int_0^\pi K_{0,3}(\pi, t)\frac{\sin\theta_n t}{\theta_n}\,dt + (-1)^{n+1}\eta_n\pi + o(\eta_n).$$

In this formula, the first three summands compose $\Delta_0(\mu_n) = 0$, and we arrive at

$$\Delta_0(\lambda_n) = \eta_n\pi\big((-1)^{n+1} + o(1)\big). \tag{16}$$

Proceeding analogously, based on (4), we also obtain the following asymptotics:

$$\left.\begin{aligned} S_a(t, \lambda_n) &= S_a(t, \mu_n) + O(n^{-1}\eta_n), \quad & S_a'(t, \lambda_n) &= S_a'(t, \mu_n) + O(\eta_n), \\ C_a(t, \lambda_n) &= C_a(t, \mu_n) + O(\eta_n), \quad & C_a'(t, \lambda_n) &= C_a'(t, \mu_n) + O(n\eta_n) \end{aligned}\right\} \tag{17}$$

uniformly on $t \in [0, \pi]$. Using (17), we have

$$A_n := S_a(\pi, \lambda_n)\int_0^a W'(0, t, \lambda_n)q(t)\,dt =$$
$$= \left(S_a(\pi, \mu_n) + O\left(\frac{\eta_n}{n}\right)\right)\left(-\int_0^a g_n(t)q(t)\,dt + O(\eta_n)\right).$$

Since $g_n(t) = -\cos\theta_n t + O(n^{-1})$, by the Riemann – Lebesgue lemma, $\int_0^a g_n(t)q(t)\,dt = o(1)$. Using also that $S_a(\pi, \mu_n) = O(n^{-1})$, we obtain

$$A_n = -S_a(\pi, \mu_n)\int_0^a g_n(t)q(t)\,dt + o(\eta_n). \tag{18}$$

Analogously, we have

$$B_n := S_a'(0, \lambda_n)\int_a^\pi W(\pi, t, \lambda_n)q(t)\,dt = S_a'(0, \mu_n)\int_a^\pi W(\pi, t, \mu_n)q(t)\,dt + o(\eta_n).$$

Relation (9) yields that $S_a'(0, \mu_n)W(\pi, t, \mu_n) = -S_a(\pi, \mu_n)g_n(t)$, and

$$B_n = -S_a(\pi, \mu_n)\int_a^\pi g_n(t)q(t)\,dt + o(\eta_n). \tag{19}$$

Combining (16), (18), and (19) with (8) in $\lambda = \rho_n^2$, we obtain

$$0 = ((-1)^n\pi + o(1))\eta_n + S(\pi, \mu_n)\xi_n, \quad \{\xi_n\}_{n\geqslant 1} \in \ell_2.$$

For $n \geqslant K$, the value η_n is the unique solution of this equation, which leads to $\eta_n = a_n\nu_n n^{-1}$ with $\{\nu_n\}_{n\geqslant K} \in \ell_2$, and to (14).

II. From the definition, it follows that

$$a_n = a_{n+1} = \ldots = a_{n+k_n-1} = 0.$$

Then, by formula (11), μ_n is a zero of $\Delta(\lambda)$ of multiplicity not less than k_n. This means that μ_n occurs in the spectrum $\{\lambda_n\}_{n\geqslant 1}$ at least k_n times, and (15) holds up to a numeration. □

In what follows, we can assume that the numeration of $\{\lambda_n\}_{n\geqslant 1}$ satisfies (15). Denote

$$\Omega = \{n + \nu \colon n \in \mathcal{S}, \quad \nu = \overline{0, k_n - 1}\}, \quad \overline{\Omega} = \mathbb{N} \setminus \Omega. \tag{20}$$

Formula (15) yields that the part of the spectrum $\{\lambda_n\}_{n\in\Omega}$ does not depend on q, i.e. we have the degeneration condition. Each unique eigenvalue $\lambda_n = \mu_n$ in this part occurs k_n times, which restricts its multiplicity to be not less than k_n. Note that (15) follows from (14) when $n \geqslant K$ since $k_n \leqslant 1$. In [22], condition (15) was not required because, for $p = 0$, we can take $K = 1$.

Now, we unify conditions (14) and (15) into one formula. Introduce the values

$$b_{n+\nu} = \begin{cases} a_{n+\nu}, & \nu = \overline{0, p_n - 1}, \\ 1, & \nu = \overline{p_n, m_n - 1}, \end{cases} \qquad p_n := \max(1, k_n), \quad n \in \mathcal{S}.$$

Then, formulae (14) and (15) are particular cases of the following relation:

$$\lambda_n = \mu_n + b_n \varkappa_n, \quad n \geqslant 1, \quad \{\varkappa_n\}_{n\geqslant 1} \in \ell_2. \tag{21}$$

It differs from (14) and (15) only by a finite number of formulae for $n = k + \nu < K$, $k \in \mathcal{S}$, $\nu \in \overline{k_n, m_n - 1}$ with $b_n \neq 0$, being non-restrictive. For this reason, (21) is equivalent to (14) unified with (15). In particular, $n \in \Omega$ if and only if $b_n = 0$.

Note that $b_n = O(1)$, then the following asymptotics is weaker than (21):

$$\lambda_n = \rho_n^2, \quad \rho_n = n - \frac{\alpha+\beta}{2} + \frac{\omega}{\pi n} + \frac{\nu_n}{n}, \quad \{\nu_n\}_{n\geqslant 1} \in \ell_2. \tag{22}$$

By the standard approach involving Hadamard's factorization theorem (see, e.g., [4]), one can prove that

$$\Delta(\lambda) = (-1)^{\alpha} \pi^{\delta_{\alpha,\beta}} \prod_{k=1}^{\infty} \frac{\lambda_k - \lambda}{c_k}, \quad c_k := \begin{cases} \left(k - \frac{\alpha+\beta}{2}\right)^2, & k \geqslant 1 \text{ or } \alpha+\beta < 2, \\ 1, & k = 1,\ \alpha = \beta = 1. \end{cases} \tag{23}$$

Thus, by the spectrum we can uniquely reconstruct the characteristic function $\Delta(\lambda)$.

3. Main results

First, we obtain the necessary and sufficient conditions on the spectrum.

Theorem 2. *For an arbitrary sequence $\{\lambda_n\}_{n\geqslant 1}$ of complex numbers to be the spectrum of boundary value problem* (1), (2) *with some $q \in L_2(0,\pi)$, it is necessary and sufficient to satisfy* (21).

For the proof, we need the following lemma.

Lemma 2. *Let $\Delta(\lambda)$ be constructed via* (23), *where arbitrary values $\{\lambda_n\}_{n\geqslant 1}$ satisfy asymptotics* (22). *Then, the following representation holds:*

$$\Delta(\lambda) = \begin{cases} \rho^{2\alpha}\left(\dfrac{\sin\rho\pi}{\rho} - \dfrac{\cos\rho\pi}{\rho^2}\omega + \displaystyle\int_0^{\pi} \frac{\cos\rho t}{\rho^2} W(t)\, dt\right), & \alpha = \beta, \\ (-1)^{\alpha}\left(\cos\rho\pi + \dfrac{\sin\rho\pi}{\rho}\omega + \displaystyle\int_0^{\pi} \frac{\sin\rho t}{\rho} W(t)\, dt\right), & \alpha \neq \beta, \end{cases} \tag{24}$$

where $W \in L_2(0,\pi)$.

For $\alpha = \beta = 0$, the statement of the lemma easily follows from Lemma 3.3 in [6] after integration in parts. For the other combinations of α and β, the needed statements are proved by analogous computations.

Proof of Theorem 2. The necessity part was proved in the previous section. Let us prove the sufficiency part. Construct the function $\Delta(\lambda)$ via formula (23) using the given numbers $\{\lambda_n\}_{n\geqslant 1}$. Condition (21) yields asymptotics (22), and, by Lemma 2, $\Delta(\lambda)$ has the form (24).

Now, we should find a function q such that its coefficients $\xi_k = \int_0^\pi q(t) g_k(t)\, dt$ satisfy (11). For every $n \in \mathcal{S}$, relation (11) can be considered as a system of m_n linear equations with respect to the vector $[\xi_{n+\nu}]_{\nu=0}^{m_n-1}$:

$$\begin{cases} a_n \xi_{n+m_n-1} = n^{2-\alpha-\beta} \Delta(\mu_n), \\ a_n \xi_{n+m_n-2} + a_{n+1} \xi_{n+m_n-1} = n^{2-\alpha-\beta} \dfrac{\Delta'(\mu_n)}{1!}, \\ \dots \\ a_n \xi_n + a_{n+1} \xi_{n+1} + \ldots + a_{n+m_n-1} \xi_{n+m_n-1} = n^{2-\alpha-\beta} \dfrac{\Delta^{(m_n-1)}(\mu_n)}{(m_n-1)!}. \end{cases} \tag{25}$$

By (21), we have $\Delta(\mu_n) = \Delta'(\mu_n) = \ldots = \Delta^{(k_n-1)}(\mu_n) = 0$. This along with (15) yields that the first k_n rows in the system (25) turn trivial identities and that arbitrary values of $\xi_{n+\nu}$, $\nu = \overline{0, k_n - 1}$, satisfy this system. If $k_n < m_n$, then $a_{n+k_n} \neq 0$, and the rest $\xi_{n+\nu}$ are uniquely determined by subsequent application of the following formulae:

$$\left.\begin{aligned} \xi_{n+m_n-1} &= n^{2-\alpha-\beta} \frac{\Delta^{(k_n)}(\mu_n)}{a_{n+k_n} k_n!}, \\ \xi_{n+m_n-\nu} &= \frac{1}{a_{n+k_n}} \left(n^{2-\alpha-\beta} \frac{\Delta^{(k_n+\nu-1)}(\mu_n)}{(k_n+\nu-1)!} - \sum_{\eta=0}^{\nu-1} a_{n+k_n+\eta} \xi_{n+m_n-\nu+\eta} \right), \ \nu = \overline{2, m_n - k_n}. \end{aligned}\right\} \tag{26}$$

Remind that for a sufficiently large $n \geqslant K$, we have $m_n = 1$, and ξ_n either can be arbitrary (if $k_n = 1$) or it is computed via the first formula in (26) (if $k_n = 0$).

Thus, we arrive at that the part of the coefficients $\{\xi_k\}_{k \in \overline{\Omega}}$ is uniquely determined by $\{\lambda_n\}_{n \geqslant 1}$, while $\{\xi_k\}_{k \in \Omega}$ can be arbitrary (for the definition of Ω and $\overline{\Omega}$, see (20)). Applying the scheme from the proof of Theorem 1 in [22], using representation (24), we obtain that $\{n^{2-\alpha-\beta} \Delta(\mu_n) a_n^{-1}\}_{n \in \overline{\Omega} \cap \mathcal{S}} \in \ell_2$, and $\{\xi_k\}_{k \in \overline{\Omega}} \in \ell_2$. Choose arbitrary coefficients $\{\xi_k\}_{k \in \Omega} \in \ell_2$. Then, there exists $q \in L_2(0, \pi)$ such that its coefficients with respect to the basis $\{g_k(t)\}_{k \geqslant 1}$ are $\{\xi_k\}_{k \geqslant 1}$.

Consider boundary value problems (1) and (2) with such potential q. Let $\Delta_*(\lambda)$ be the characteristic function of this boundary value problem. Then, by construction,

$$F(\lambda) = \frac{\Delta_*(\lambda) - \Delta(\lambda)}{\Delta_0(\lambda)}$$

is an entire function. Representations (8) and (24) along with (4) yield asymptotics

$$\Delta_*(\lambda) - \Delta(\lambda) = O\left(\rho^{\alpha+\beta-2} e^{\tau\pi}\right), \quad \rho^2 = \lambda, \ \tau = |\mathrm{Im}\rho|. \tag{27}$$

Using (4), we also arrive at

$$\Delta_0(\lambda) = \begin{cases} (-1)^\alpha \left(\cos \rho\pi + O\Big(\dfrac{e^{\tau\pi}}{\rho}\Big) \right), & \alpha \neq \beta, \\ \rho^{2\alpha-1} \left(\sin \rho\pi + O\Big(\dfrac{e^{\tau\pi}}{\rho}\Big) \right), & \alpha = \beta. \end{cases}$$

Consider arbitrary $\delta \in (0, \frac{1}{4})$. For a sufficiently large $|\lambda| \geqslant N_\delta$, analogously to (13), one can prove that

$$|\Delta_0(\lambda)| \geqslant C_\delta |\rho|^{\alpha+\beta-1} e^{\tau\pi}, \quad \rho \in G_\delta = \left\{ z \in \mathbb{C}\colon \left| z - n + \frac{\alpha+\beta}{2} \right| \geqslant \delta, \ n \in \mathbb{Z} \right\}, \tag{28}$$

where $C_\delta > 0$. Using (27) and (28), we arrive at $F(\lambda) = o(1)$ in G_δ. By the maximum modulus principle and Liouville's theorem, $F(\lambda) = 0$. Thus, the function $\Delta(\lambda)$ is the characteristic function of the boundary value problem (1), (2) with the considered potential q, and $\{\lambda_n\}_{n\geqslant 1}$ is its spectrum. □

From the proof of Theorem 2, it follows that the potentials q corresponding to the same spectrum have the same coefficients $\{\xi_n\}_{n\in\overline{\Omega}}$, while for $n \in \Omega$, the coefficients ξ_n may differ. At the same time, by Proposition 1, the mapping $q \mapsto \{\xi_n\}_{n\geqslant 1}$ is a one-to-one correspondence between $L_2(0,\pi)$ and ℓ_2. Thus, for a fixed spectrum $\{\lambda_n\}_{n\geqslant 1}$, one can construct the set of all iso-spectral potentials q varying $\{\xi_n\}_{n\in\Omega} \in \ell_2$ or find a unique q setting additionally $\{\xi_n\}_{n\in\Omega} \in \ell_2$. In the latter case, we obtain a uniqueness theorem.

Theorem 3. *Let $\{\tilde\lambda_n\}_{n\geqslant 1}$ be the spectrum of boundary value problem* (1), (2) *with a potential $\tilde q \in L_2(0,\pi)$, while $\tilde\xi_n = \int_0^\pi \tilde q(t) g_n(t)\,dt$, $n \in \Omega$. If $\{\lambda_n\}_{n\geqslant 1} = \{\tilde\lambda_n\}_{n\geqslant 1}$ and $\xi_n = \tilde\xi_n$ for $n \in \Omega$, then $q = \tilde q$.*

Since the proof of Theorem 2 is constructive, we have the following algorithm for recovering q given $\{\lambda_n\}_{n\geqslant 1}$ and $\{\xi_n\}_{n\in\Omega}$.

Algorithm 1. To recover the potential q, one should:

1. Construct $\Delta(\lambda)$ via formula (23).
2. For $n \in \mathcal{S}$, by formula (26), compute the unknown coefficients $\xi_{n+k_n}, \dots, \xi_{n+m_n-1}$.
3. Find $q = \sum_{n=1}^{\infty} \xi_n f_n$, where $\{f_n\}_{n\geqslant 1}$ is the basis biorthonormal to $\{\overline{g}_n\}_{n\geqslant 1}$ in $L_2(0,\pi)$.

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