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Article

Shear waves in a nonlinear elastic cylindrical shell

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Abstract. Asymptotic integration methods have been used to model the propagation of a shear wave beam along a nonlinear-elastic cylindrical shell of the Sanders – Koiter model. The shell is assumed to be made of a material characterized by a cubic dependence between stress and strain intensities, and the dimensionless parameters of thinness and physical nonlinearity are considered to have the same order of smallness. The multiscale expansion method is used, which makes it possible to determine the wave propagation speed from the equations of the linear approximation, and in the first essentially nonlinear approximation, to obtain a nonlinear quasi-hyperbolic equation for the main term of the expansion of the shear displacement component. The derived equation is a cubically nonlinear modification of the Lin – Reisner – Tsien equation modeling unsteady near-sonic gas flow and can be transformed into the modified Khokhlov – Zabolotskaya equation used to describe narrow beams in acoustics. The solution of the derived equation is found in the form of a single harmonic with slowly changing complex amplitude, since in deformable media with cubic nonlinearity the effect of self-induced wave essentially prevails over the effect of generation of higher harmonics. As a result, a perturbed nonlinear Schrödinger equation of defocusing type is obtained for the complex amplitude, for which there is no possibility of modulation instability development. In terms of the elliptic Jacobi function, an exact physically consistent solution, periodic along the dimensionless circumferential coordinate, is constructed.
Keywords: nonlinear elastic cylindrical shell, shear waves, asymptotic integration, nonlinear Schrödinger equation
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Сдвиговые волны в нелинейно-упругой цилиндрической оболочке

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Аннотация. Методами асимптотического интегрирования проведено моделирование распространения пучка сдвиговых волн вдоль образующей нелинейно-упругой цилиндрической оболочки модели Сандерса – Койтера. Считается, что оболочка изготовлена из материала, характеризующегося кубической зависимостью между интенсивностями напряжений и деформаций, а безразмерные параметры тонкостенности и физической нелинейности являются величинами одного порядка малости. Используется разновидность метода многомасштабных разложений, позволяющая из уравнений линейного приближения определить скорость распространения волны, а в первом существенно нелинейном приближении — получить разрешающее нелинейное квазигиперболическое уравнение для главного члена разложения сдвиговой компоненты смещения. Выведенное уравнение представляет собой кубически нелинейную модификацию уравнения Линя – Рейснера – Цзяна, моделирующего нестационарное околозвуковое течение газа, и может быть преобразовано в модифицированное уравнение Заболотской – Хохлова, используемое для описания узких пучков в акустике. Решение выведенного уравнения отыскивается в виде одной гармоники с медленно меняющейся комплексной амплитудой, поскольку в деформируемых средах с кубической нелинейностью эффект самовоздействия волны существенно преобладает над эффектом генерации высших гармоник. В результате для комплексной амплитуды получено возмущенное нелинейное уравнение Шредингера дефокусирующего типа, для которого отсутствует возможность развития модуляционной неустойчивости. В терминах эллиптической функции Якоби построено точное физически состоятельное решение, периодическое по безразмерной окружной координате.

Ключевые слова: нелинейно-упругая цилиндрическая оболочка, сдвиговые волны, асимптотическое интегрирование, нелинейное уравнение Шредингера

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Introduction

Currently, there is an increasing interest in the study of nonlinear shear waves in systems of various physical natures in relation to problems of acoustic diagnostics and non-destructive testing. Biomedical applications of shear waves are discussed in [1, 2]. The possibility of diagnosing pathologies and the functional state of the muscular system is shown, due to the fact that the speed of these waves in the muscles is much lower than the speed of longitudinal waves and the formation of higher symmetry. Corresponding acoustic methods have great promise in the diagnosis of neuralgic pathologies, as well as in gerontology, sports, and space medicine. In [3], a method for measuring shear elasticity using focused ultrasound radiation pressure known as SWEI (Shear Wave Elasticity Imaging) was proposed. The development of this technique called supersonic shear imaging (SSI) is discussed in [4,5]. The results of observing a shear wave excited by focused ultrasound in a rubber-like medium are given in [6]. A theoretical model has been constructed that allows one to calculate the characteristics of the generated shear wave depending on the parameters of the medium and the initial longitudinal wave. In [7], plane nonlinear shear waves in a medium with memory are considered. Model equations with cubic nonlinearity are derived and analyzed based on the Duffing equation. The dissertation [8] is devoted to the study of the propagation of linear and nonlinear shear waves in viscoelastic media. Classes of exact wave solutions for shear waves in special viscous media were constructed in [9]. In [10], the effects of cubic nonlinearity in shear wave beams of different polarizations are analyzed analytically and numerically. Solitary and compact shear waves are studied in [11]. The possibility of generating the second harmonic for shear waves in an elastoplastic medium was demonstrated in [12]. Nonlinear shear waves in a solid with a microstructure are studied in [13]. In [14], shear waves in a nonlinear elastic body are analytically modeled. The propagation of a beam of shear waves in a hereditary medium in the quasi-optical approximation is considered in [15]. In [16], the propagation of shear solitons in an elastic plate is modeled. In [17–20], quasi-hyperbolic and evolutionary equations that can be reduced to integrable ones are derived. Thus, the possibility of generation and conditions for the existence of shear solitons are demonstrated. Nonlinear longitudinal and shear stationary deformation waves in a gradient-elastic medium are considered in [17]. It is shown that stationary shear waves are described by the Duffing equation. The propagation of nonlinear shear waves in a granular medium is considered in [19]. It is shown that low-frequency soliton-like disturbances are described by the Boussinesq equation. In [20], the modeling of solitary shear waves in a granular medium led to the perturbed sine-Gordon equation for the first time. It is shown that the speed of a solitary wave is always less than the speed of a transverse seismic wave. In [21], the influence of material heterogeneity on the evolution of a Riemann shear wave is studied. The dependence of the characteristic breaking distance of the Riemann wave on the values of the corresponding elastic moduli has been revealed.

Problems of nonlinear wave dynamics of cylindrical shells, in contrast to similar problems for rods and plates [22], have been studied to a lesser extent to date. The reason is that in rods and plates, tangential and normal displacements are separated in a linear approximation and are coupled only due to nonlinearity, which simplifies the use of asymptotic integration procedures [23] of systems of initial equations. In shells, due to curvature, longitudinal, circumferential and normal displacements are already connected in a linear approximation, and the division of wave movements into longitudinal, shear (torsional), and bending becomes, to a certain extent, conditional.

Depending on which component of the displacements predominates, we have to talk about longitudinal-flexural, flexural-longitudinal, flexural-shear, etc. waves. Nonlinear axisymmetric waves of the longitudinal-flexural type are considered in [24–27]. Flexural-longitudinal waves are studied in [28]. In these works, classes of exact soliton-like and periodic solutions were constructed and questions of their physical realizability were discussed. This article is devoted to modeling the propagation of a predominantly shear wave along the generatrix of a cylindrical shell.

1. Derivation of the resolving equation

We will carry out further analysis based on the Sanders – Koiter model of a cylindrical shell [29, 30]. Geometrically linear equations of motion of a shell element in forces and moments have the form:

$$\begin{gathered}\frac{\partial N_x}{\partial x}+\frac{\partial N_{xy}}{\partial y}+\frac{1}{2R}\frac{\partial M_{xy}}{\partial y}-\frac{\gamma h}{g}\frac{\partial^2 u}{\partial t^2}=0,\\ \frac{\partial N_{xy}}{\partial x}+\frac{\partial N_y}{\partial y}-\frac{3}{2R}\frac{\partial M_{xy}}{\partial x}-\frac{1}{R}\frac{\partial M_y}{\partial y}-\frac{\gamma h}{g}\frac{\partial^2 v}{\partial t^2}=0,\\ \frac{\partial^2 M_x}{\partial x^2}+2\frac{\partial^2 M_{xy}}{\partial x\partial y}+\frac{\partial^2 M_y}{\partial y^2}+\frac{1}{R}N_y-\frac{\gamma h}{g}\frac{\partial^2 w}{\partial t^2}=0,\end{gathered} \tag{1}$$

where x, y are longitudinal and circumferential coordinates, u, v, w are movement of the middle surface of the shell element in the longitudinal, circumferential and radial directions, respectively, N_x, M_x are normal force and bending moment in a section perpendicular to the axis Ox, N_y, M_y are normal force and bending moment in a section perpendicular to the axis Oy, N_{xy}, M_{xy} are shear force and torque in a plane section Oxy, h, R, γ are thickness, radius of curvature and surface density of the shell, g is acceleration of gravity and t is time.

We will assume that the shell is made of a nonlinear elastic material characterized by a cubic relationship between the intensities of stress and strain [31]

$$\sigma_i = E_0\varepsilon_i - m\varepsilon_i^3,$$

where E_0 is the initial elastic modulus and m is the material constant determined experimentally [32].

The relationship between forces and moments with deformations and changes in curvature of the middle surface is assumed to be as follows:

$$\begin{gathered} N_x = \frac{\left(E_0 - me_i^2\right)h}{1-\mu^2}\left(e_x + \mu e_y\right), \\ N_y = \frac{\left(E_0 - me_i^2\right)h}{1-\mu^2}\left(e_y + \mu e_x\right), \\ N_{xy} = \frac{\left(E_0 - me_i^2\right)h}{2\left(1+\mu\right)}e_{xy}, \end{gathered} \tag{2}$$

$$\begin{gathered} M_x = \frac{\left(E_0 - me_i^2\right)h^3}{12\left(1-\mu^2\right)}\left(\kappa_x + \mu\kappa_y\right) - \frac{2mFh^3}{27\left(1-\mu^2\right)\left(1-\mu\right)^2}\left(e_x + \mu e_y\right), \\ M_y = \frac{\left(E_0 - me_i^2\right)h^3}{12\left(1-\mu^2\right)}\left(\kappa_y + \mu\kappa_x\right) - \frac{2mFh^3}{27\left(1-\mu^2\right)\left(1-\mu\right)^2}\left(e_y + \mu e_x\right), \\ M_{xy} = \frac{\left(E_0 - me_i^2\right)h^3}{12\left(1-\mu^2\right)}\cdot\frac{1-\mu}{2}\cdot\kappa_{xy} - \frac{2mFh^3}{27\left(1-\mu^2\right)\left(1-\mu\right)^2}\cdot\frac{1-\mu}{2}\cdot e_{xy}, \end{gathered} \tag{3}$$

where

$$e_i^2 = \frac{4}{9}\left(e_x^2 - e_xe_y + e_y^2 + \frac{\mu}{\left(1-\mu\right)^2}\left(e_x + e_y\right)^2 + \frac{1}{3}e_{xy}^2\right)$$

represents the square of the intensity of deformations of the middle surface, μ is Poisson's ratio and

$$F = \left(1 - \mu + \mu^2\right)\left(\kappa_xe_x + \kappa_ye_y\right) - \frac{1}{2}\left(1 - 4\mu + \mu^2\right)\left(\kappa_ye_x + \kappa_xe_y\right) + \frac{3}{4}\kappa_{xy}e_{xy}(1-\mu)^2.$$

The components of changes in curvature and deformation in the linearized Sanders – Koiter model are related to the components of displacements by the following relations:

$$\begin{gathered} e_x = \frac{\partial u}{\partial x}, \quad \kappa_x = -\frac{\partial^2 w}{\partial x^2}, \\ e_y = \frac{\partial v}{\partial y} - \frac{1}{R}w, \quad \kappa_y = -\left[\frac{\partial^2 w}{\partial y^2} + \frac{1}{R}\frac{\partial v}{\partial y}\right], \\ e_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \kappa_{xy} = -2\left[\frac{\partial^2 w}{\partial x\partial y} + \frac{1}{4R}\left(3\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right]. \end{gathered} \tag{4}$$

Substituting (4) into (2) and (3), and then the resulting relations into (1), we obtain a system of equations of motion of the shell element in displacements

$$\begin{gathered} \frac{\partial^2 u}{\partial x^2} + \frac{1-\mu}{2}\left(1 + \underline{\frac{h^2}{48R^2}}\right)\frac{\partial^2 u}{\partial y^2} + \frac{1+\mu}{2}\left(1 - \underline{\frac{(1-\mu)h^2}{16(1+\mu)R^2}}\right)\frac{\partial^2 v}{\partial x\partial y} - \\ -\frac{\mu}{R}\frac{\partial w}{\partial x} - \underline{\frac{(1-\mu)h^2}{24R}\frac{\partial^3 w}{\partial x\partial y^2}} - \frac{\gamma\left(1-\mu^2\right)}{gE_0}\frac{\partial^2 u}{\partial t^2} + f_1 = 0, \end{gathered} \tag{5}$$

$$\left(1+\underline{\frac{h^2}{12R^2}}\right)\frac{\partial^2 v}{\partial y^2}+\left(1+\underline{\frac{3h^2}{16R^2}}\right)\frac{1-\mu}{2}\frac{\partial^2 v}{\partial x^2}+\left(1-\underline{\frac{(1-\mu)\,h^2}{16\,(1+\mu)\,R^2}}\right)\frac{1+\mu}{2}\frac{\partial^2 u}{\partial x\partial y}-$$
$$-\frac{1}{R}\frac{\partial w}{\partial y}+\underline{\frac{h^2}{12R}\frac{\partial^3 w}{\partial y^3}}+\underline{\frac{(3-\mu)\,h^2}{24R}\frac{\partial^3 w}{\partial x^2\partial y}}-\frac{\gamma\left(1-\mu^2\right)}{gE_0}\frac{\partial^2 v}{\partial t^2}+f_2=0, \tag{6}$$

$$-\frac{h^2}{12}\nabla^4 w+\frac{\mu}{R}\frac{\partial u}{\partial x}+\frac{1}{R}\frac{\partial v}{\partial y}-\frac{1}{R^2}w+\underline{\frac{(1-\mu)\,h^2}{24R}\frac{\partial^3 u}{\partial x\partial y^2}}-$$
$$-\underline{\frac{h^2}{12R}\frac{\partial^3 v}{\partial y^3}}-\underline{\frac{(3-\mu)\,h^2}{24R}\frac{\partial^3 v}{\partial x^2\partial y}}-\frac{\gamma\left(1-\mu^2\right)}{gE_0}\frac{\partial^2 w}{\partial t^2}+f_3=0, \tag{7}$$

in which all nonlinear terms are reduced to f_1, f_2 and f_3. The resulting system differs from a similar system of equations of motion in the classical theory of Kirchhoff – Love shells [32] by the presence of underlined terms of higher orders of smallness. Taking them into account during further asymptotic integration, as will be shown below, leads to a new form of the resolving equation.

Let us introduce dimensionless independent and dependent variables into consideration using the formulas:

$$X=\frac{x}{l},\quad Y=\frac{y}{R},\quad T=\sqrt{\frac{E_0 g}{\gamma\,(1-\mu^2)}}\cdot\frac{t}{l},$$
$$U=\frac{u}{R},\quad V=\frac{v}{R},\quad W=\frac{w}{h}. \tag{8}$$

Substituting (8) into the equations of motion (5)–(7) reveals in the latter small parameters of nonlinearity and thin-walledness $\frac{R}{l}$ and $\frac{h}{R}$, which we will consider to be quantities of the same order of smallness with the ratio of the physical nonlinearity parameters:

$$\frac{R}{l}=\frac{h}{R}=\frac{E_0}{m}=\varepsilon. \tag{9}$$

This assumption is typical for problems of asymptotic analysis of long-wave perturbations [24–26]. According to (9), the dispersion parameter $\frac{h}{l}$ is a value of the order of ε^2.

Assuming that a beam of shear waves propagates along the generatrix of the shell, we introduce into consideration new independent variables and expansions of dependent variables in powers of the small parameter ε:

$$\chi=\varepsilon^2 X,\quad \eta=\varepsilon^2 Y,\quad \tau=T-\frac{X}{C},$$
$$U=\varepsilon^{3/2}U_0+\varepsilon^{5/2}U_1,\quad V=\varepsilon^{1/2}V_0+\varepsilon^{5/2}V_1,\quad W=\varepsilon^{3/2}W_0+\varepsilon^{5/2}W_1, \tag{10}$$

where C is the yet unknown wave propagation speed. Taking into account (8)–(10), in leading order in powers of the small parameter, the system of equations (5)–(7) takes the form

$$\frac{\mu}{C}\frac{\partial W_0}{\partial\tau}+\left(\frac{1}{C^2}-1\right)\frac{\partial^2 U_0}{\partial\tau^2}-\frac{(1+\mu)}{2C}\frac{\partial^2 V_0}{\partial\tau\,\partial\eta}=0, \tag{11}$$

$$\left(2C^2+\mu-1\right)\frac{\partial^2 V_0}{\partial\tau^2}=0, \tag{12}$$

$$\frac{\mu}{C}\frac{\partial U_0}{\partial\tau}+W_0-\frac{\partial V_0}{\partial\eta}=0. \tag{13}$$

From (12) the dimensionless wave propagation speed

$$C=\pm\sqrt{\frac{1-\mu}{2}}.$$

is determined. The equation (6) in the first essentially nonlinear approximation has the form

$$\frac{\partial^2 V_0}{\partial\tau\,\partial\chi} - \frac{3\sqrt{2-2\mu}}{32\,(1-\mu)}\frac{\partial^2 V_0}{\partial\tau^2} - \frac{1}{\sqrt{2-2\mu}}\frac{\partial^2 V_0}{\partial\eta^2} + \frac{1+\mu}{2\,(1-\mu)}\frac{\partial^2 U_0}{\partial\tau\,\partial\eta} + \\ + \frac{1}{\sqrt{2-2\mu}}\frac{\partial W_0}{\partial\eta} + \frac{\sqrt{2-2\mu}}{(1-\mu)^2}\left(\frac{\partial V_0}{\partial\tau}\right)^2\frac{\partial^2 V_0}{\partial\tau^2} = 0. \quad (14)$$

Using (11), (13) to express the terms with the functions U_0 and W_0 in terms of V_0, we give the equation (14) its final form

$$\frac{\partial^2 V_0}{\partial\tau\,\partial\chi} - c_1\frac{\partial^2 V_0}{\partial\tau^2} + c_2\frac{\partial^2 V_0}{\partial\eta^2} + c_3\left(\frac{\partial V_0}{\partial\tau}\right)^2\frac{\partial^2 V_0}{\partial\tau^2} = 0, \quad (15)$$

where

$$c_1 = \frac{3}{16\sqrt{2-2\mu}}, \quad c_2 = \frac{\sqrt{1-\mu}}{2\sqrt{2}\,(1+2\mu)}, \quad c_3 = \frac{\sqrt{2}}{(1-\mu)\sqrt{1-\mu}}.$$

2. Discussion of the properties of equation (15) and construction of its periodic solutions

An equation similar to (15) was derived in [15] for a quasi-plane shear wave in a medium with memory. The cubically nonlinear equation (15) contains an additional term with coefficient c_1 arising due to the underlined terms in the system (5)–(7). Such an equation with quadratic nonlinearity formally coincides with the Lin – Reisner – Tsien (LRT) equation modelling unsteady near-sonic gas flow [33]. Therefore, by analogy with the classical and modified Korteweg – de Vries equations, (15) can be called the modified LRT equation. At the same time, for the function $f = \frac{\partial V_0}{\partial\tau}$, this equation takes the form of the modified Khokhlov – Zabolotskaya (KZ) equation [33]

$$\frac{\partial}{\partial\tau}\left(\frac{\partial f}{\partial\chi} - c_1\frac{\partial f}{\partial\tau} + c_3 f^2\frac{\partial f}{\partial\tau}\right) = -c_2\frac{\partial^2 f}{\partial\eta^2},$$

the second summand in the left-hand side of which is eliminated by passing to the corresponding traveling wave variable. It is known that in nonlinear acoustics the abbreviation KZ is more often used, while in aerodynamics the abbreviation LRT is more common [33].

With variable transformation

$$V_0 = \sqrt{\frac{c_1}{c_3}}\Phi, \quad t = c_1\chi, \quad \xi = c_1\chi + \tau, \quad \zeta = \sqrt{\frac{c_1}{c_2}}\eta$$

the equation (15) takes the compact form

$$\frac{\partial^2\Phi}{\partial t\,\partial\xi} + \left(\frac{\partial\Phi}{\partial\xi}\right)^2\frac{\partial^2\Phi}{\partial\xi^2} + \frac{\partial^2\Phi}{\partial\zeta^2} = 0, \quad (16)$$

typical for problems of gas dynamics and acoustics. It is obvious that equation (16) has a large set of symmetries, and its invariant group analysis is a separate task for further research.

When analyzing nonlinear wave processes in cubically nonlinear deformable media, the experimentally established fact of a significant predominance of the phenomenon of wave self-action over the effect of generation of higher harmonics is usually used [34, 35]. This makes it possible to find a solution to equations of the type (15) in the form of a single harmonic with a slowly varying complex amplitude

$$V_0 = A\,(\chi,\eta)\exp\left(-i\omega\tau\right) + c.c. \quad (17)$$

Substituting (17) into (15), we obtain the perturbed nonlinear Schrödinger equation (NSE) for the amplitude A

$$i\omega\frac{\partial A}{\partial\chi} - c_1\omega^2 A - c_2\frac{\partial^2 A}{\partial\eta^2} + c_3\omega^4 A^2 |A| = 0. \qquad (18)$$

To find an exact periodic solution to the NSE (18) we make the substitution $A(\chi,\eta) = a(\eta)\exp(i\omega\chi)$, after which for the amplitude $a(\eta)$ we obtain the Duffing equation

$$\frac{d^2 a}{d\eta^2} + d_1 a + d_2 a^3 = 0,$$

where

$$d_1 = \frac{c_1+1}{c_2}\omega^2, \quad d_2 = -\frac{c_3}{c_2}\omega^4.$$

A physically consistent solution, depending on the dimensionless circumferential coordinate η, must be periodic in η. This condition is met by the general solution of the Duffing equation in terms of the Jacobi elliptic function

$$a = C_1 \mathrm{sn}\left[g_1\left(g_2\eta + C_2\right), g_3\right],$$

in which C_1, C_2 are constants of integration and

$$g_1 = \sqrt{\frac{2d_1}{2d_1 + d_2 - C_1^2 d_2}}, \quad g_2 = \frac{1}{2}\sqrt{4d_1 + 2d_2}, \quad g_3 = C_1\frac{\sqrt{-\left(2d_1+d_2\right)d_2}}{2d_1+d_2},$$

under condition $g_3 \neq \pm 1$.

Note that in equation (18) the coefficients in front of the dispersion and nonlinear terms have opposite signs, that is, a defocusing version of the NSE is obtained. It is known [36] that in this case, the development of modulation instability is impossible, that is, a spatially homogeneous solution is stable, as well as solutions in the form of periodic stationary waves.

Conclusion

It has been established that when modelling the propagation of a shear wave beam along a nonlinear-elastic cylindrical shell, a quasi-hyperbolic equation with cubic nonlinearity, generalizing the LRT and KZ equations, is formed. Thus, the connection between the problems of nonlinear wave dynamics of deformable systems, near-sonic gas dynamics, and nonlinear acoustics is demonstrated. It is shown that cubic nonlinearity allows to transform the derived equation into the NSE of defocusing type, in which it is impossible to develop the modulation instability of periodic wave solutions.

References

1. Rudenko O., Sarvazyan A. Wave biomechanics of skeletal muscle. *Journal of the Acoustical Society of America*, 2006, vol. 120, art. 3270. https://doi.org/10.1121/1.4777047
2. Rudenko O. V. Nonlinear waves: Some biomedical applications. *Physics-Uspekhi*, 2007, vol. 50, iss. 4, art. 359. https://doi.org/10.1070/PU2007v050n04ABEH006236
3. Sarvazyan A. P., Rudenko O. V., Swanson S. D., Fowlkes J. B., Emelianov S. Y. Shear wave elasticity imaging: A new ultrasonic technology of medical diagnostics. *Ultrasound in Medicine and Biology*, 1998, vol. 24, iss. 9, pp. 1419–1435. https://doi.org/10.1016/S0301-5629(98)00110-0
4. Rénier M., Gennisson J.-L., Tanter M., Catheline S., Barrière C., Royer D., Fink M. 7B-2 nonlinear shear elastic moduli in quasi-incompressible soft solids. *2007 IEEE Ultrasonics Symposium Proceedings*. New York, NY, 2007, pp. 554–557. https://doi.org/10.1109/ultsym.2007.144
5. Gennisson J.-L. New parameters in shear wave elastography in vivo. In: *Mécanique pour le vivant. Identification et modélisation du comportement des tissus biologiques humains et animaux. Avancées et perspectives*. Colloque National, du 18 au 22 janvier 2016. Available at: https://mecamat.ensma.fr/Aussois/2016/DOCUMENT/TexteGenisson.pdf (accessed May 10, 2024).

6. Andreev V. G., Dmitriev V. N., Pishchal'nikov Yu. A., Rudenko O. V., Sapozhnikov O. A., Sarvazyan A. P. Observation of shear waves excited by focused ultrasound in a rubber-like medium. *Acoustical Physics*, 1997, vol. 43, iss. 2, pp. 123–128. EDN: LEFXVP
7. Cormack J. M., Hamilton M. F. Plane nonlinear shear waves in relaxing media. *The Journal of the Acoustical Society of America*, 2018, vol. 143, iss. 2, pp. 1035–1048. https://doi.org/10.1121/1.5023394
8. Lindley B. S. *Linear and nonlinear shear wave propagation in viscoelastic media*. University of North Carolina, Chapel Hill, 2008. 78 p. Available at: https://cdr.lib.unc.edu/downloads/z029p543p (accessed May 10, 2024).
9. Rajagopal K. R., Saccomandi G. Shear waves in a class of nonlinear viscoelastic solids. *The Quarterly Journal of Mechanics and Applied Mathematics*, 2003, vol. 56, iss. 2, pp. 311–326. https://doi.org/10.1093/qjmam/56.2.311
10. Wochner M. S., Hamilton M. F., Ilinskii Y. A., Zabolotskaya E. A. Cubic nonlinearity in shear wave beams with different polarizations. *Journal of the Acoustical Society of America*, 2008, vol. 123, iss. 5, pp. 2488–2495. https://doi.org/10.1121/1.2890739
11. Destrade M., Saccomandi G. Solitary and compact-like shear waves in the bulk of solids. *Physical Review E*, 2006, vol. 73, iss. 6, art. 065604. https://doi.org/10.1103/PhysRevE.73.065604
12. Doronin A. M., Erofeev V. I. A generation of second harmonic of shear wave in elastoplastic media. *Letters on Materials*, 2016, vol. 6, iss. 2, pp. 102–104 (in Russian). https://doi.org/10.22226/2410-3535-2016-2-102-104
13. Erofeev V. I. Propagation of nonlinear shear waves in a solid with microstructure. *International Applied Mechanics*, 1993, vol. 29, pp. 262–266. https://doi.org/10.1007/BF00847023
14. Raskin I. G., Erofeyev V. I. Propagation of sound shear waves in the nonlinear elastic solids. *Soviet Applied Mechanics*, 1991, vol. 27, iss. 1, pp. 127–129 (in Russian).
15. Potapov A. I., Soldatov I. N. Quasioptical approximation for a bundle of shear waves in a nonlinear hereditary medium. *Prikladnaya Mekhanika i Tekhnicheskaya Fizika*, 1986, vol. 27, iss. 1, pp. 144–147 (in Russian).
16. Kivshar Yu. S., Syrkin E. S. Shear solitons in an elastic plate. *Akusticheskiy Zhurnal*, 1991, vol. 37, iss. 1, pp. 104–109 (in Russian).
17. Erofeev V. I., Sheshenina O. A. Nonlinear longitudinal and shear stationary deformation waves in a gradient-elastic medium. *Matematicheskoe modelirovanie system i protsessov* [Mathematical Modeling of Systems and Processes], 2007, iss. 15, pp. 15–27 (in Russian).
18. Erofeev V. I., Kolesov D. A., Sandalov V. M. Demodulation of a shear wave in a nonlinear plate resting on an elastic foundation with the parameters changing following the running wave law. *Problemy prochnosti i plastichnosti* [Problems of Strength and Plasticity], 2013, vol. 75, iss. 4, pp. 268–272 (in Russian). https://doi.org/10.32326/1814-9146-2013-75-4-268-272
19. Bogdanov A. N., Skvortsov A. T. Nonlinear shear waves in a granular medium. *Akusticheskiy Zhurnal*, 1992, vol. 38, iss. 3, pp. 408–412 (in Russian).
20. Bykov V. G. Solitary shear waves in a granular medium. *Acoustical Physics*, 1999, vol. 45, iss. 2, pp. 138–142.
21. Erofeyev V. I., Sharabanova A. V. Riemann shift waves in material with properties depending on the stress state type. *Problemy Mashinostroeniya i Nadezhnosti Mashin*, 2004, vol. 1, pp. 20–23 (in Russian). EDN: OWBWPP
22. Erofeyev V. I., Kajaev V. V., Semerikova N. P. *Volny v sterzhnyakh. Dispersiya. Dissipatsiya. Nelineynost'* [Waves in rods. Dispersion. Dissipation. Nonlinearity]. Moscow, Fizmatlit, 2002. 208 p. (in Russian).
23. Kaplunov J. D., Kossovich L. Yu., Nolde E. V. *Dynamics of thin walled elastic bodies*. San Diego, Academic Press, 1998. 226 p. https://doi.org/10.1016/C2009-0-20923-8, EDN: WNSAFB
24. Zemlyanukhin A. I., Andrianov I. V., Bochkarev A. V., Mogilevich L. I. The generalized Schamel equation in nonlinear wave dynamics of cylindrical shells. *Nonlinear Dynamics*, 2019, vol. 98, pp. 185–194. https://doi.org/10.1007/s11071-019-05181-5
25. Zemlyanukhin A. I., Bochkarev A. V., Andrianov I. V., Erofeev V. I. The Schamel – Ostrovsky equation in nonlinear wave dynamics of cylindrical shells. *Journal of Sound and Vibration*, 2021, vol. 491, art. 115752. https://doi.org/10.1016/j.jsv.2020.115752
26. Zemlyanukhin A. I., Bochkarev A. V., Artamonov N. A. Physically admissible and inadmissible exact localized solutions in problems of nonlinear wave dynamics of cylindrical shells. *Russian Journal of Nonlinear Dynamics*, 2024, vol. 20, iss. 2, pp. 219–229. https://doi.org/10.20537/nd240602

27. Zemlyanukhin A. I., Bochkarev A. V., Ratushny A. V., Chernenko A. V. Generalized model of nonlinear elastic foundation and longitudinal waves in cylindrical shells. *Izvestiya of Saratov University. Mathematics. Mechanics. Informatics*, 2022, vol. 22, iss. 2, pp. 196–204. https://doi.org/10.18500/1816-9791-2022-22-2-196-204
28. Zemlyanukhin A. I., Bochkarev A. V. Axisymmetric nonlinear modulated waves in a cylindrical shell. *Acoustical Physics*, 2018, vol. 64, pp. 408–414. https://doi.org/10.1134/S1063771018040139
29. Yamaki N., Simitses G. J. Elastic stability of circular cylindrical shells. *Journal of Applied Mechanics*, 1985, vol. 52, iss. 2, pp. 501–502. https://doi.org/10.1115/1.3169089
30. Amabili M. A comparison of shell theories for large-amplitude vibrations of circular cylindrical shells: Lagrangian approach. *Journal of Sound and Vibration*, 2003, vol. 264, iss. 5, pp. 1091–1125. https://doi.org/10.1016/S0022-460X(02)01385-8
31. Lukash P. A. *Osnovy nelineinoy stroitel'noy mekhaniki* [Fundamentals of nonlinear structural mechanics]. Moscow, Stroyizdat, 1978. 208 p. (in Russian).
32. Volmir A. *The nonlinear dynamics of plates and shells*. Foreign Tech. Div., Wright-Patterson AFB, Ohio, 1974. 450 p.
33. Rudenko O. V. The 40th anniversary of the Khokhlov – Zabolotskaya equation. *Acoustical Physics*, 2010, vol. 56, pp. 457–466. https://doi.org/10.1134/S1063771010040093
34. Zarembo L. K., Krasil'nikov V. A. *Vvedenie v nelineynuyu akustiku: Zvukovye i ul'trazvukovye volny bol'shoy intensivnosti* [Introduction to nonlinear acoustics: High intensity sound and ultrasonic waves]. Moscow, Nauka, 1966. 520 p. (in Russian).
35. Zarembo L. K., Krasil'nikov V. A. Nonlinear phenomena in the propagation of elastic waves in solids. *Soviet Physics Uspekhi*, 1971, vol. 13, iss. 6, pp. 778–797. https://doi.org/10.1070/PU1971v013n06ABEH004281
36. Ryskin N. M., Trubetskov D. I. *Nelineynye volny* [Nonlinear waves]. Moscow, Lenand, 2017. 312 p. (in Russian).

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