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Article

On structure of isomorphisms of universal graphic automata

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Abstract. Automata theory is one of the branches of mathematical cybernetics, that studies information transducers that arise in many applied problems. The major objective of automata theory is to develop methods by which one can describe and analyze the dynamic behavior of discrete systems. Depending on study tasks, automata are considered, for which the set of states and the set of output signals are equipped with additional mathematical structure preserved by transition and output functions of automata. We investigate automata over graphs and call them graphic automata. Universal graphic automaton $\mathrm{Atm}(G, H)$ is a universally attractive object in the category of such automata. The semigroup of input signals of the automaton is $S(G, H) = \mathrm{End}\ G \times \mathrm{Hom}(G, H)$. It can be considered as a derivative algebraic system of the mathematical object $\mathrm{Atm}(G, H)$, which contains useful information about the initial automaton. It is common knowledge that properties of the semigroup are interconnected with properties of the algebraic structure of the automaton. Hence, it is possible to study universal graphic automata by researching their input signal semigroups. Earlier the authors proved that a wide class of such kind of automata are determined up to isomorphism by their input signal semigroups. In this paper, we investigate a connection between isomorphisms of universal graphic automata and isomorphisms of their components — semigroups of input signals and graphs of states and output signals.

Keywords: automata, graph, semigroup, isomorphism, automorphism

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О структуре изоморфизмов универсальных графовых автоматов

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Аннотация. Теория автоматов — один из разделов математической кибернетики, изучающий преобразователи информации, возникающие во многих прикладных задачах. Основная цель теории автоматов — разработка методов, с помощью которых можно описывать и анализировать динамическое поведение дискретных систем. В зависимости от исследуемых задач рассматриваются автоматы, у которых множество состояний и множество выходных сигналов наделены дополнительной математической структурой, согласованной с функциями переходов и выходов автомата. Мы исследуем автоматы над графами и называем их графовыми автоматами. Универсальный графовый автомат $\mathrm{Atm}(G, H)$ является универсально притягивающим объектом в категории таких автоматов. Полугруппа входных сигналов такого автомата имеет вид $S(G, H) = \mathrm{End}\ G \times \mathrm{Hom}(G, H)$. Её можно рассматривать как производную алгебраическую систему математического объекта $\mathrm{Atm}(G, H)$, содержащую полезную информацию об исходном автомате. Известно, что свойства полугруппы взаимосвязаны со свойствами алгебраической структуры автомата. Следовательно, можно изучать универсальные графовые автоматы, исследуя их полугруппы входных сигналов. Ранее авторы доказали, что широкий класс таких автоматов определяется (с точностью до изоморфизма) своими полугруппами входных сигналов. В данной работе исследуется связь изоморфизмов универсальных графовых автоматов с изоморфизмами их компонент — полугрупп входных сигналов и графов состояний и выходных сигналов.

Ключевые слова: автомат, граф, полугруппа, изоморфизм, автоморфизм

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Introduction

One of the main topics of modern algebra is an investigation of mathematical objects by studying derived algebraic systems associated with these objects. Various algebraic systems are considered the initial mathematical objects, and the automorphism groups, the endomorphism semigroups, the lattices of subsystems of algebraic systems, and others are considered as the derived algebraic systems. For the automorphism groups of algebraic systems, the endomorphism semigroups of graphs, the endomorphism rings of modules, and other derived algebraic systems, these questions were very successfully investigated by B. I. Plotkin [1], A. G. Pinus [2, 3], L. M. Gluskin [4, 5], Yu. M. Vazhenin [6, 7], A. V. Mikhalev [8], and other algebraists.

It is also of interest to study structured automata in the categories [9], that is automata, in which the sets of states and output signals are equipped with mathematical structures from a category $\mathbf{K}$, and transition and output functions are morphisms of this category. A set of input signals is usually equipped with an associative operation, which makes it an object of the semigroup category. The study of such automata belongs to the direction described above: in this case, the initial object is an automaton, and the derived system is the semigroup of its input signals, which are considered transformations of the set of the automaton states.

In this paper, we consider automata over the graph category $\mathbf{Gr}$, which is called graphic automata. As follows from [9], in the category of graphic automata with a graph of states G_1 and a graph of output signals G_2, there is the universal attracting object $\mathrm{Atm}(G_1, G_2)$, which is called universal graphic automaton over the graphs G_1, G_2. The semigroup of input signals of such automaton $S = \mathrm{End}\ G_1 \times \mathrm{Hom}(G_1, G_2)$ is regarded as the derived algebraic system of the automaton $\mathrm{Atm}(G_1, G_2)$. In the paper [10], authors investigated the problem of definability of such automata by their input signal semigroups: it is shown that the universal graphic automata over many reflexive graphs are completely determined (up to isomorphism and graph duality) by their semigroups of input signals. In this article, for such automata, we consider the structure of their isomorphisms and groups of automorphisms. Theorem 2 shows a connection between isomorphisms of universal graphic automata and isomorphisms of the automaton components — the graph of states, the graph of output signals, and the input signal semigroup. For universal graphic automata over quasi-acyclic graphs of states and antisymmetric graphs of output signals, in Theorem 3, we obtain the isomorphism structure description of the input signal semigroups of automata, and in Theorem 4, we obtain the structure description of the automorphism group of automata.

1. Basic notions

We assume that the reader is familiar with basic notions of the semigroup theory [11], the automata theory [9], and the graph theory [12]. Let us briefly unify the basic notations used in this work.

From now on, by a graph, we mean a directed graph. For a graph $G = (X, \rho)$, an edge $(x, y) \in \rho$ is called proper if $(y, x) \notin \rho$. A graph is called quasi-acyclic if each of its proper edges does not belong to any cycle. Acyclic graphs, quasi-order graphs, and many others are examples of quasi-acyclic graphs. A quasi-acyclic graph will be called trivial if it has no proper edges, and nontrivial otherwise. For a graph $G = (X, \rho)$ the graph $\tilde{G} = (X, \rho^{-1})$ is called the dual graph of G.

In what follows, under connectivity components of a graph, we keep in mind weak connectivity components. A graph is called connected if it has only one connectivity component.

An anti-isomorphism of a graph $G_1 = (X_1, \rho_1)$ onto a graph $G_2 = (X_2, \rho_2)$ is an isomorphism of the graph G_1 onto the graph $\tilde{G}_2$ dual to G_2. An anti-automorphism of a graph $G = (X, \rho)$ is an isomorphism of the graph G onto its dual graph $\tilde{G}$.

A semigroup automaton is an algebraic system $A = (X_1, S, X_2, \star, \diamond)$ consisting of a set of states X_1, an input signal semigroup $(S, \cdot)$, a set of output signals X_2, a transition function $\star : X_1 \times S \to X_1$, and an output function $\diamond : X_1 \times S \to X_2$, satisfying

$$x \star (s_1 \cdot s_2) = (x \star s_1) \star s_2,$$
$$x \diamond (s_1 \cdot s_2) = (x \star s_1) \diamond s_2$$

for every $x \in X_1$, $s_1, s_2 \in S$.

A semigroup automaton $A = (X_1, S, X_2, \star, \diamond)$ is called graphic if its set of states X_1 and set of output signals X_2 are equipped with structures of graphs $G_1 = (X_1, \rho_1)$, $G_2 = (X_2, \rho_2)$ such that for every input signal $s \in S$ a transition function $\delta_s = x \star s$ $(x \in X_1)$ is an endomorphism of G_1 and an output function $\lambda_s = x \diamond s$ $(x \in X_1)$ is a homomorphism of G_1 in G_2. In this case, we denote the automaton by $A = (G_1, S, G_2, \star, \diamond)$. For any graphs $G_1 = (X_1, \rho_1)$, $G_2 = (X_2, \rho_2)$ the graphic automaton $\mathrm{Atm}(G_1, G_2) = (G_1, S, G_2, \star, \diamond)$ with the input signal semigroup $S = \mathrm{End}\ G_1 \times \mathrm{Hom}(G_1, G_2)$, consisting of pairs $s = (\varphi, \psi)$, $\varphi \in \mathrm{End}\ G_1$, $\psi \in \mathrm{Hom}(G_1, G_2)$, and functions $x \star s = \varphi(x)$, $x \diamond s = \psi(x)$ $(x \in X_1)$, is the universally attracting object in the category of graphic automata, that is why it is called universal graphic automaton [9].

A mapping $c_x : X \to \{x\}$ is called a constant mapping of a set X to an element x. For mappings $f : X \to Y$, $g : Y \to Z$ a composition is defined by the formula $(f \cdot g)(x) = g(f(x))$ for $x \in X$. For mappings $f : X \to Y$, $g : X \to Y$ a direct product $f \times g : X \times X \to Y \times Y$ is defined by the formula $(f \times g)(u, v) = (f(u), g(v))$. For any transformation φ of a set X, it is true that $(f \times g)(\varphi) = f^{-1} \varphi g$. We denote $f \times f = f^2$.

An isomorphism of a graphic automaton $A_1 = (G_1, S_1, G_1', \star_1, \diamond_1)$, where $G_1 = (X_1, \rho_1)$, $G_1' = (X_1', \rho_1')$, onto a graphic automaton $A_2 = (G_2, S_2, G_2', \star_2, \diamond_2)$, where $G_2 = (X_2, \rho_2)$, $G_2' = (X_2', \rho_2')$, is an ordered triple $\gamma = (f, h, g)$, consisting of isomorphisms $f : G_1 \to G_2$,

$h : S_1 \to S_2$, $g : G'_1 \to G'_2$ such that for any $x \in X_1$, $s, t \in S_1$ the following conditions hold:

$$h(s \cdot t) = h(s) \cdot h(t),$$
$$f(x \star_1 s) = f(x) \star_2 h(s),$$
$$g(x \diamond_1 s) = f(x) \diamond_2 h(s).$$

An isomorphism of an automaton $A = (G, S, G', \star, \diamond)$ onto itself is called an automorphism of the automaton A. The set of all automorphisms of A with the composition forms the automorphism group Aut A of the automaton A.

2. Preparatory phase

To solve the main issue, we use some auxiliary results obtained in [10].

Lemma 1. *Let $G_1 = (X_1, \rho_1)$, $G_2 = (X_2, \rho_2)$ be reflexive graphs. Then the following statements are true for the semigroup $S = \mathrm{End}\ G_1 \times \mathrm{Hom}(G_1, G_2)$:*

1) *an element $s \in S$ is a right zero of the semigroup S if and only if there exist $a \in X_1$, $b \in X_2$ such that $s = (c_a, c_b)$;*
2) *an element $s \in S$ is a left identity of the semigroup S if and only if $s = (\Delta_X, \psi)$ for some $\psi \in \mathrm{Hom}(G_1, G_2)$.*

For graphs G_1, G_2, we denote by $Z(G_1, G_2)$ the set of all right zeros of the semigroup $S = \mathrm{End}\ G_1 \times \mathrm{Hom}(G_1, G_2)$, by $U(G_1, G_2)$ — the set of all left identities of the semigroup S. It is clear that the set $Z(G_1, G_2)$ is defined in the semigroup S by the predicate $M(x) = (\forall y)(y \cdot x = x)$ of the semigroup theory, and the set $U(G_1, G_2)$ is defined in the semigroup S by the predicate $N(x) = (\forall y)(x \cdot y = y)$ of the semigroup theory.

Lemma 2. *Let $G_1 = (X_1, \rho_1)$, $G_2 = (X_2, \rho_2)$ be reflexive graphs. Then the formula of the semigroup theory*

$$E(x, y) = M(x) \wedge M(y) \wedge (\forall e)(N(e) \Longrightarrow x \cdot e = y \cdot e)$$

defines a binary relation ε on the semigroup $S = \mathrm{End}\ G_1 \times \mathrm{Hom}(G_1, G_2)$, such that the following statements hold:

1) *ε is an equivalence on the set $Z(G_1, G_2)$ such that for any elements $s_1, s_2 \in Z(G_1, G_2)$ the condition $s_1 \equiv_\varepsilon s_2$ is valid if and only if $s_1 = (c_a, c_u)$, $s_2 = (c_a, c_v)$ for some $a \in X_1$, $u, v \in X_2$;*
2) *for any right zero $s = (c_a, c_b)$ of the semigroup S, the equivalence class $\varepsilon(s) = \{(c_a, c_u) | u \in X_2\}$.*

By analogy with Lemma 1 in [13], we can obtain the following result.

Lemma 3. *Let $G = (X, \rho)$, $H = (Y, \sigma)$ be reflexive graphs, $v \in X$, $(x, y) \in \sigma$. A mapping $f : X \to Y$ defined for $u \in X$ by the formula*

$$f(u) = \begin{cases} y, & \text{if there is a path from } v \text{ to } u, \\ x, & \text{otherwise}, \end{cases}$$

is an homomorphism of G_1 in G_2.

3. Main results

The following result describes the relationship between isomorphisms of the input signal semigroup of a universal graphic automaton and isomorphisms of its graph of states and its graph of output signals.

Theorem 1. *Let $G_i = (X_i, \rho_i), G'_i = (X'_i, \rho'_i)$ be reflexive graphs ($i = 1, 2$), the graph G_1 has an edge that does not belong to any cycle, and let $\mathrm{Atm}(G_1, G'_1)$, $\mathrm{Atm}(G_2, G'_2)$ be the universal graphic automata with the input signal semigroups $S_i = \mathrm{End}\ G_i \times \mathrm{Hom}(G_i, G'_i)$ ($i = 1, 2$), $h : S_1 \to S_2$ is an isomorphism of the semigroup S_1 onto the semigroup S_2. Then there exist isomorphisms f, g_a ($a \in X_1$) of graphs G_1, G'_1 onto graphs G_2, G'_2 respectively or onto their dual graphs $\tilde{G}_2$, $\tilde{G}'_2$ respectively, such that for any pair $(\varphi, \psi) \in S_1$ the equality*

$$h(\varphi, \psi) = (f^2(\varphi), \psi^{\varphi}) \tag{1}$$

holds, where $\psi^{\varphi}(f(a)) = g_{\varphi(a)}(\psi(a))$ for all $a \in X_1$.

Proof. Consider reflexive graphs $G_i = (X_i, \rho_i), G'_i = (X'_i, \rho'_i)$ ($i = 1, 2$), such that the graph G_1 has an edge that does not belong to any cycle, and an isomorphism h of the semigroup $S_1 = \mathrm{End}\ G_1 \times \mathrm{Hom}(G_1, G'_1)$ onto the semigroup $S_2 = \mathrm{End}\ G_2 \times \mathrm{Hom}(G_2, G'_2)$.

It is common knowledge that every semigroup isomorphism preserves the satisfiability of formulas of the elementary semigroup theory. Hence, the isomorphism h preserves the satisfiability of the formulas $M(x)$, $N(x)$, $E(x, y)$. Therefore, the isomorphism h maps the set of all right zeros $Z(G_1, G'_1)$ of the semigroup S_1 onto the set of all right zeros $Z(G_2, G'_2)$ of the semigroup S_2, the set of all left identities $U(G_1, G'_1)$ of the semigroup S_1 onto the set of all left identities $U(G_2, G'_2)$ of the semigroup S_2. According to Lemma 2, the Cartesian product h^2 maps the equivalence $\varepsilon_1 = \varepsilon_{(G_1, G'_1)}$ (defined in the semigroup S_1 by the formula $E(x, y)$) onto the equivalence $\varepsilon_2 = \varepsilon_{(G_2, G'_2)}$ (defined in the semigroup S_2 by the formula $E(x, y)$).

According to item 1 of Lemma 1, for any $a \in X_1$, $b \in X'_1$, there are elements $d \in X_2$, $e \in X'_2$ such that $h(c_a, c_b) = (c_d, c_e)$. The isomorphism h maps the equivalence class $\varepsilon_1(c_a, c_b)$ to the equivalence class $\varepsilon_2(c_d, c_e)$. Therefore, formulas $f(a) = d$, $g_a(b) = e$ define mappings $f : X_1 \to X_2$, $g_a : X'_1 \to X'_2$ ($a \in X_1$) such that

$$h(c_a, c_b) = (c_{f(a)}, c_{g_a(b)}).$$

It is easy to see that mappings $f : G_1 \to G_2$, $g_a : G'_1 \to G'_2$ ($a \in X_1$) are bijections.

Let $(\varphi, \psi) \in S_1$, $a \in X_1$ and $\varphi(a) = d$, $\psi(a) = e$. Then

$$(c_a, c_b) \cdot (\varphi, \psi) = (c_a\varphi, c_a\psi) = (c_{\varphi(a)}, c_{\psi(a)}) = (c_d, c_e).$$

Since h is an isomorphism of S_1 onto S_2, the equality

$$h(c_a, c_b) \cdot h(\varphi, \psi) = h(c_d, c_e)$$

holds. We denote $h(\varphi, \psi) = (\varphi', \psi')$. According to the construction of mappings $f : X_1 \to X_2$, $g_a : X'_1 \to X'_2$ ($a \in X_1$), we obtain

$$h(c_a, c_b) = (c_{f(a)}, c_{g_a(b)}),\ \ h(c_d, c_e) = (c_{f(d)}, c_{g_d(e)}).$$

Then

$$(c_{f(a)}, c_{g_a(b)}) \cdot (\varphi', \psi') = (c_{f(d)}, c_{g_d(e)}),$$
$$(c_{f(a)}\varphi', c_{f(a)}\psi') = (c_{f(d)}, c_{g_d(e)}),$$
$$(c_{\varphi'(f(a))}, c_{\psi'(f(a))}) = (c_{f(d)}, c_{g_d(e)}),$$

and, hence, $\varphi'(f(a)) = f(d) = f(\varphi(a))$, $\psi'(f(a)) = g_d(e) = g_{\varphi(a)}(\psi(a))$. Therefore,

$$\varphi' = \{(f(a), f(\varphi(a)))|a \in X_1\} = f^2(\varphi),$$
$$\psi' = \{(f(a), g_{\varphi(a)}(\psi(a)))|a \in X_1\} = \psi^{\varphi},$$

where $\psi^{\varphi}(f(a)) = g_{\varphi(a)}(\psi(a))$ for all $a \in X_1$.

Hence, for any pair $(\varphi, \psi) \in S_1$, the equality (1) holds.

It is easy to verify that the Cartesian product $f^2 : \mathrm{End}\, G_1 \to \mathrm{End}\, G_2$ is a bijection. Moreover, for any $\varphi_1, \varphi_2 \in \mathrm{End}\, G_1$ the following equalities holds:

$$f^2(\varphi_1\varphi_2) = f^{-1}\varphi_1\varphi_2 f = f^{-1}\varphi_1\Delta_{X_1}\varphi_2 f = f^{-1}\varphi_1 f f^{-1}\varphi_2 f = f^2(\varphi_1)f^2(\varphi_2).$$

It follows that f^2 is an isomorphism of $\mathrm{End}\, G_1$ onto $\mathrm{End}\, G_2$. According to theorem condition, the graph G_1 has an edge $(u_0, v_0) \in \rho_1$ that does not belong to any cycle. On the strength of Yu. M. Vazhenin's result [7], the mapping f is an isomorphism or an anti-isomorphism of the graph $G_1 = (X_1, \rho_1)$ onto the graph $G_2 = (X_2, \rho_2)$.

Suppose that f is an isomorphism of the graph G_1 onto the graph G_2. It is necessary to prove that, for any $a \in X_1$, the mapping g_a is an isomorphism of the graph G'_1 onto the graph G'_2.

Let $(x_0, y_0) \in \rho'_1$ holds for some $x_0, y_0 \in X'_1$. The mapping $\psi : G_1 \to G'_1$, defined for $u \in X_1$ by the formula

$$\psi(u) = \begin{cases} y_0, & \text{if there exists a path from } v_0 \text{ to } u, \\ x_0, & \text{otherwise,} \end{cases}$$

is a homomorphism of the graph G_1 into the graph G'_1 due to Lemma 3, and $\psi^2(u_0, v_0) = (x_0, y_0)$. It means $(c_a, \psi) \in S_1$, and from (1) it follows that $h(c_a, \psi) = (c_{f(a)}, \psi^{c_a})$, $\psi^{c_a} \in \mathrm{Hom}(G_2, G'_2)$. On the other hand, $\psi^{c_a}(f(x)) = g_{c_a(x)}(\psi(x)) = g_a(\psi(x))$. Then for $x = u_0$ we get $\psi^{c_a}(f(u_0)) = g_a(\psi(u_0)) = g_a(x_0)$, and for $x = v_0$ we get $\psi^{c_a}(f(v_0)) = g_a(\psi(v_0)) = g_a(y_0)$. Hence ψ^{c_a} maps $(f(u_0), f(v_0))$ into $(g_a(x_0), g_a(y_0))$. Since $(f(u_0), f(v_0)) \in \rho_2$ and ψ^{c_a} is a homomorphism of the graph G_2 into the graph G'_2, then $(g_a(x_0), g_a(y_0)) \in \rho'_2$. Thus, $g_a \in \mathrm{Hom}(G'_1, G'_2)$.

Conversely, let the condition $(x'_0, y'_0) \in \rho'_2$ hold for some $x'_0, y'_0 \in X'_2$. Then from Lemma 3 it follows that for some homomorphism $\psi_1 \in \mathrm{Hom}(G_2, G'_2)$ the equation $\psi_1(f(u_0), f(v_0)) = (x'_0, y'_0)$ holds. Hence $(c_{f(a)}, \psi_1) \in S_2$, and there exists such pair $(\varphi, \psi) \in S_1$ that $h(\varphi, \psi) = (c_{f(a)}, \psi_1)$. Thus, due to the equation (1), we have that $c_{f(a)} = f^2(\varphi) = f^{-1}\varphi f$, $\psi_1 = \psi^{\varphi}$. Then

$$\varphi = (ff^{-1})\varphi(ff^{-1}) = fc_{f(a)}f^{-1} = f(f^{-1}c_a f)f^{-1} = (ff^{-1})c_a(ff^{-1}) = c_a.$$

As a result we get

$$x'_0 = \psi_1(f(u_0)) = \psi^{c_a}(f(u_0)) = g_a(\psi(u_0)),\ \ \psi(u_0) = g_a^{-1}(x'_0),$$
$$y'_0 = \psi_1(f(v_0)) = \psi^{c_a}(f(v_0)) = g_a(\psi(v_0)),\ \ \psi(v_0) = g_a^{-1}(y'_0)$$

and $(g_a^{-1}(x'_0), g_a^{-1}(y'_0)) \in \rho'_1$, therefore $g_a^{-1} \in \mathrm{Hom}(G'_2, G'_1)$. By this means g_a $(a \in X_1)$ is a family of isomorphisms of the graph G'_1 onto the graph G'_2.

Analogously if f is an isomorphism of the graph G_1 onto the graph $\tilde{G}_2$, then all mappings g_a $(a \in X_1)$ are isomorphisms of the graph G'_1 onto the graph $\tilde{G'_2}$, because in this case the condition $(u_0, v_0) \in \rho_1$ is equivalent to $(f(v_0), f(u_0)) \in \rho_2$, the condition $(x_0, y_0) \in \rho'_1$ is equivalent to $(g_a(y_0), g_a(x_0)) \in \rho'_2$. □

The following result shows the connection between isomorphisms of universal graphic automata and their components.

Theorem 2. *Let $G_i = (X_i, \rho_i), G'_i = (X'_i, \rho'_i)$ be graphs $(i = 1, 2)$ and f be an isomorphism of G_1 onto G_2, g be an isomorphism of G'_1 onto G'_2. The ordered triple of mappings $\gamma = (f, h, g)$ is an isomorphism of the universal graphic automaton $A_1 = \mathrm{Atm}(G_1, G'_1)$ with the input signal semigroup $S_1 = \mathrm{End}\, G_1 \times \mathrm{Hom}(G_1, G'_1)$ onto the universal graphic automaton $A_2 = \mathrm{Atm}(G_2, G'_2)$ with the input signal semigroup $S_2 = \mathrm{End}\, G_2 \times \mathrm{Hom}(G_2, G'_2)$ if and only if the mapping $h : S_1 \to S_2$ is defined for any $(\varphi, \psi) \in S_1$ by the formula $h(\varphi, \psi) = (f^2(\varphi), (f \times g)(\psi))$.*

Proof. *Necessity.* Let $\gamma = (f, h, g)$ be an isomorphism of the automaton A_1 onto A_2. Then for any $x \in X_1$, $s = (\varphi_1, \psi_1) \in S_1$ the following conditions hold:

$$f(x \star_1 s) = f(x) \star_2 h(s), \quad g(x \diamond_1 s) = f(x) \diamond_2 h(s).$$

Then for the image $h(s) = (\varphi_2, \psi_2) \in S_2$ for all $x \in X_1$ it follows

$$f(\varphi_1(x)) = \varphi_2(f(x)), \quad g(\psi_1(x)) = \psi_2(f(x)).$$

Hence, $\varphi_1 f = f\varphi_2$, $\psi_1 g = f\psi_2$, and it follows $\varphi_2 = f^{-1}\varphi_1 f = f^2(\varphi_1)$, $\psi_2 = f^{-1}\psi_1 g = (f \times g)(\psi_1)$. Therefore, $h(\varphi_1, \psi_1) = (f^2(\varphi_1), (f \times g)(\psi_1))$.

Sufficiency. Let isomorphisms f of G_1 onto G_2 and g of G'_1 onto G'_2 define a mapping $h : S_1 \to S_2$ by the formula $h(\varphi, \psi) = (f^2(\varphi), (f \times g)(\psi))$ for all $(\varphi, \psi) \in S_1$. Let $(\varphi, \psi) \in S_1$. By the definition $\varphi \in \mathrm{End}\ G_1$, $\psi \in \mathrm{Hom}(G_1, G'_1)$, hence $f^2(\varphi) = f^{-1}\varphi f \in \mathrm{End}\ G_2$, $(f \times g)(\psi) = f^{-1}\psi g \in \mathrm{Hom}(G_2, G'_2)$. It follows that $h(\varphi, \psi) \in S_2$.

It is easy to verify that $h : S_1 \to S_2$ is a bijective mapping. Extra to this, for any $s_1 = (\varphi_1, \psi_1)$, $s_2 = (\varphi_2, \psi_2)$ from S_1 the following equalities hold:

$$\begin{aligned} h(s_1 \cdot s_2) &= (f^2(\varphi_1\varphi_2), (f \times g)(\varphi_1\psi_2)) = (f^{-1}\varphi_1\varphi_2 f, f^{-1}\varphi_1\psi_2 g) = \\ &= (f^{-1}\varphi_1 f f^{-1}\varphi_2 f, f^{-1}\varphi_1 f f^{-1}\psi_2 g) = (f^2(\varphi_1)f^2(\varphi_2), f^2(\varphi_1)(f \times g)(\psi_2)) = \\ &= (f^2(\varphi_1), (f \times g)(\psi_1)) \cdot (f^2(\varphi_2), (f \times g)(\psi_2)) = h(s_1) \cdot h(s_2). \end{aligned}$$

Thus, h is an isomorphism of S_1 onto S_2.

Let $x \in X_1$, $(\varphi, \psi) \in S_1$. The following equalities hold:

$$\begin{aligned} f(x) \star_2 h(s) &= f(x) \star_2 (f^2(\varphi), (f \times g)(\psi)) = f^2(\varphi)(f(x)) = \\ &= (f^{-1}\varphi f)(f(x)) = f(\varphi(f^{-1}(f(x)))) = f(\varphi(x)) = f(x \star_1 s), \\ f(x) \diamond_2 h(s) &= f(x) \diamond_2 (f^2(\varphi), (f \times g)(\psi)) = ((f \times g)(\psi))(f(x)) = \\ &= (f^{-1}\psi g)(f(x)) = g(\psi(f^{-1}(f(x)))) = g(\psi(x)) = g(x \diamond_1 s). \end{aligned}$$

Therefore, the ordered triple $\gamma = (f, h, g)$ is an isomorphism of the automaton A_1 onto the automaton A_2. □

Theorem 2 implies that for automata $A_1 = \mathrm{Atm}(G_1, G'_1)$, $A_2 = \mathrm{Atm}(G_2, G'_2)$ any isomorphism $\gamma = (f, h, g)$ of A_1 onto A_2 is completely determined by a pair of isomorphisms of state graphs and output signal graphs. On the other hand, the set of isomorphisms of semigroups of input signals of such automata is much larger than the set of isomorphisms of automata. It is demonstrated by the following example.

Let $G = (X, \rho)$ be a graph with connectivity components $X_1, X_2, \ldots, X_n, \ldots$ and $G' = (\mathbb{Z}, \leqslant)$. For any $n \in \mathbb{N}$, we define a transformation g_n of the graph G' in such a manner: $g_n(z) = z + n$ $(z \in \mathbb{Z})$. All transformations g_n are automorphisms of the graph G'. Consider a universal graphic automaton $\mathrm{Atm}(G, G')$ with the semigroup of input signals $S = \mathrm{End}\ G \times \mathrm{Hom}(G, G')$. For any pair $(\varphi, \psi) \in S$, we set $h(\varphi, \psi) = (\varphi, \psi^\varphi)$, where $\psi^\varphi(a) = g_n(\psi(a))$ for all $a \in X$, satisfying the condition $\varphi(a) \in X_n$. It is clear that the mapping h is an automorphism of the semigroup S, but it cannot be the second component of any automorphism of the automaton $\mathrm{Atm}(G, G')$.

The following example shows that for universal graphic automata not all isomorphisms of the state graphs and families of isomorphisms of the output signal graphs define isomorphisms of the semigroups of input signals.

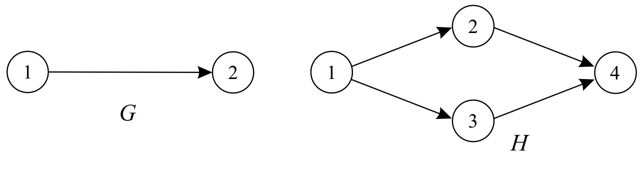

Figure. Graphs G and H

Let $G = (X_G, \rho_G)$, $H = (X_H, \rho_H)$ be reflexive graphs pictured in the Figure (loops are not shown), $\mathrm{Atm}(G, H)$ is a universal graphic automaton with the semigroup of input signals $S = \mathrm{End}\, G \times \mathrm{Hom}(G, H)$. Consider the following two automorphisms of the graph H: $g_1 = \Delta_H$ — the identity automorphism of the graph H, $g_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}$. For each $(\varphi, \psi) \in S$, we set $h(\varphi, \psi) = (\varphi, \psi^\varphi)$, where $\psi^\varphi(x) = g_{\varphi(x)}(\psi(x))$ for all $x \in X_G$. Then, for the identity endomorphism $\varphi = \Delta_G$ and for the constant mapping ψ of the set X_G to the vertex 2 of the graph H, the condition $(\varphi, \psi) \in S$ holds, but $h(\varphi, \psi) \notin S$ because

$$h(\varphi, \psi) = \left(\Delta_G, \begin{pmatrix} 1 & 2 \\ g_{\varphi(1)}(\psi(1)) & g_{\varphi(2)}(\psi(2)) \end{pmatrix}\right) = \left(\Delta_G, \begin{pmatrix} 1 & 2 \\ g_1(2) & g_2(2) \end{pmatrix}\right) = \left(\Delta_G, \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}\right)$$

and

$$\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \notin \mathrm{Hom}(G, H).$$

Therefore, $h \notin \mathrm{Aut}\ S$, i.e. the identity automorphism Δ_H of the graph H and the family of automorphisms g_1, g_2 of the graph H do not define an automorphism of the automaton $\mathrm{Atm}(G, H)$.

The following result describes the structure of isomorphisms of the semigroups of input signals of universal graphic automata.

Theorem 3. *Let $G_1 = (X_1, \rho_1)$, $G_1' = (X_1', \rho_1')$, $G_2 = (X_2, \rho_2)$, $G_2' = (X_2', \rho_2')$ be reflexive graphs, besides G_1' is an antisymmetric graph, G_1 is a nontrivial quasi-acyclic graph with connectivity components $\{X_{1_i}\}$, $i \in I$, and let $\mathrm{Atm}(G_1, G_1')$, $\mathrm{Atm}(G_2, G_2')$ be the universal graphic automata with the semigroups of input signals $S_1 = \mathrm{End}\ G_1 \times \mathrm{Hom}(G_1, G_1')$ and $S_2 = \mathrm{End}\ G_2 \times \mathrm{Hom}(G_2, G_2')$ correspondingly. Then a mapping $h : S_1 \to S_2$ is an isomorphism of the semigroup S_1 onto the semigroup S_2 if and only if for some isomorphism (anti-isomorphism) $f : G_1 \to G_2$ and some family of isomorphisms (anti-isomorphisms) $g_i : G_1' \to G_2'$, $i \in I$, for all $(\varphi, \psi) \in S_1$ the mapping h is defined by the formula*

$$h(\varphi, \psi) = (f^2(\varphi), \psi^\varphi), \tag{2}$$

where $\psi^\varphi(f(a)) = g_i(\psi(a))$ for any $a \in X_1$, such that the condition $\varphi(a) \in X_{1_i}$ is satisfied for some $i \in I$.

Proof. *Necessity.* Let $h : S_1 \to S_2$ be an isomorphism of S_1 onto S_2. By Theorem 1 the isomorphism h inspires bijections $f : G_1 \to G_2$, $g_a : G_1' \to G_2'$ $(a \in X_1)$ by the formulas:

$$f(a) = b \iff (\exists y \in X_1', z \in X_2')\ h(c_a, c_y) = (c_b, c_z)\ (a \in X_1, b \in X_2),$$
$$g_a(y) = z \iff h(c_a, c_y) = (c_{f(a)}, c_z)\ (y \in X_1', z \in X_2').$$

According to the construction of the bijections f, g_a $(a \in X_1)$, we have

$$h(c_a, c_x) = (c_{f(a)}, c_{g_a(x)}),\ h^{-1}(c_{f(a)}, c_y) = (c_a, c_{g_a^{-1}}(x)). \tag{3}$$

By Theorem 1 for any pair of mappings $(\varphi, \psi) \in S_1$ equation (2) holds, the mapping f is an isomorphism (or an anti-isomorphism) of G_1 onto G_2, the family of mappings g_a are isomorphisms (or anti-isomorphisms) of G_1' onto G_2' for all $a \in X_1$.

We now show that, for adjacent vertices a, b of the graph G_1, isomorphisms g_a, g_b are equal. For definiteness, let $(a, b) \in \rho_1$ and let f be an isomorphism of G_1 onto G_2. If the edge $(a, b) \in \rho_1$ is proper, then by Lemma 1 [13], there exists an endomorphism $\varphi_1 \in \mathrm{End}\ G_1$ such that $\varphi_1(X_1) = \{a, b\}$, $\varphi_1(a) = a$, $\varphi_1(b) = b$. If the edge $(a, b) \in \rho_1$ has an opposite edge, then consider the transformation $\varphi_2 : G_1 \to G_1$, which is defined for all $u \in X_1$ by the formula

$$\varphi_2(u) = \begin{cases} a, & u = a, \\ b, & u \neq a. \end{cases}$$

Obviously $\varphi_2 \in \mathrm{End}\, G_1$. Therefore, in any case, there is an endomorphism $\varphi \in \mathrm{End}\, G_1$ such that $\varphi(X_1) = \{a, b\}$, $\varphi(a) = a$, $\varphi(b) = b$. Then for all $x \in X_1'$ we get

$$h(\varphi, c_x) = (f^2(\varphi), c_x^{\varphi}) = \left(f^2(\varphi), \begin{pmatrix} \dots & f(a) & \dots & f(b) & \dots \\ \dots & g_a(x) & \dots & g_b(x) & \dots \end{pmatrix}\right) \in S_2.$$

Since $(a, b) \in \rho_1$ and f is an isomorphism of G_1 onto G_2, then $(f(a), f(b)) \in \rho_2$, and since $c_x^{\varphi} \in \mathrm{Hom}(G_2, G_2')$, then $(g_a(x), g_b(x)) \in \rho_2'$. Similarly, using formulas (3), it is possible to show that, for $f(a), f(b) \in X_2$ and any $y \in X_2'$, the condition $(g_a^{-1}(y), g_b^{-1}(y)) \in \rho_1'$ holds. Then, for isomorphisms g_a, g_b of the graph G_1' onto the graph G_2' and for any $x \in X_1'$, we get

$$(g_a(x), g_b(x)) \in \rho_2' \Longrightarrow (g_b^{-1}(g_a(x)), g_b^{-1}(g_b(x)) \in \rho_1' \Longleftrightarrow (g_b^{-1}(g_a(x)), x) \in \rho_1',$$
$$(g_a^{-1}(g_a(x)), g_b^{-1}(g_a(x))) \in \rho_1' \Longleftrightarrow (x, g_b^{-1}(g_a(x))) \in \rho_1'.$$

Since the graph G_1' is antisymmetric, then $g_b^{-1}(g_a(x)) = x$, hence $g_a \cdot g_b^{-1} = \Delta_{X_1'}$, i.e. $g_a = g_b$.

Sufficiency. Let for an isomorphism $f : G_1 \to G_2$ and a family of isomorphisms $g_i : G_1' \to G_2'$ $(i \in I)$ a mapping $h : S_1 \to S_2$ is defined by the formula

$$h(\varphi, \psi) = (f^2(\varphi), \psi^{\varphi}),$$

where $\psi^{\varphi}(f(a)) = g_i(\psi(a))$ for all $a \in X_1$ such that the condition $\varphi(a) \in X_{1_i}$ is satisfied for some $i \in I$.

Let's check that $h(\varphi, \psi) \in S_2$ for any $(\varphi, \psi) \in S_1$. In Theorem 1 it was shown that f^2 is an isomorphism of $\mathrm{End}\, G_1$ onto $\mathrm{End}\, G_2$, hence $f^2(\varphi) \in \mathrm{End}\, G_2$.

Let $(u_2, v_2) \in \rho_2$, $u_1 = f^{-1}(u_2)$, $v_1 = f^{-1}(v_2)$. Since f is an isomorphism of G_1 onto G_2, then $(u_1, v_1) \in \rho_1$, and since $\varphi \in \mathrm{End}\, G_1$, then $(\varphi(u_1), \varphi(v_1)) \in \rho_1$ and $\varphi(u_1), \varphi(v_1)$ belong to some connectivity component X_{1_i} of the graph G_1. Since $\psi \in \mathrm{Hom}(G_1, G_1')$, we get $(\psi(u_1), \psi(v_1)) \in \rho_1'$. Then for the isomorphism g_i of the graph G_1' onto the graph G_2' we obtain that $(g_i(\psi(u_1)), g_i(\psi(v_1))) \in \rho_2'$ and, consequently, $(\psi^{\varphi}(u_2), \psi^{\varphi}(v_2)) \in \rho_2'$. This implies $\psi^{\varphi} \in \mathrm{Hom}(G_2, G_2')$ and $h(\varphi, \psi) \in S_2$.

Let us verify that the mapping h is a bijection. Let us show that the mapping h is injective: let $(\varphi_1, \psi_1), (\varphi_2, \psi_2) \in S_1$, $(\varphi_1, \psi_1) \neq (\varphi_2, \psi_2)$. If $\varphi_1 \neq \varphi_2$ then $f^2(\varphi_1) \neq f^2(\varphi_2)$, hence $h(\varphi_1, \psi_1) \neq h(\varphi_2, \psi_2)$. If $\varphi_1 = \varphi_2$, then $\psi_1 \neq \psi_2$, that is, there exists an element $a \in X_1$ such that $\psi_1(a) \neq \psi_2(a)$. Suppose the element a belongs to a connectivity component X_{1_i} of the graph G_1. Then $g_i(\psi_1(a)) \neq g_i(\psi_2(a))$ and hence $\psi_1^{\varphi}(f(a)) \neq \psi_2^{\varphi}(f(a))$, i.e. $\psi_1^{\varphi} \neq \psi_2^{\varphi}$. Thus $h(\varphi_1, \psi_1) \neq h(\varphi_2, \psi_2)$ and h is injective.

Let us show that the mapping h is surjective: let $(\varphi_2, \psi_2) \in S_2$. We define mappings $\varphi_1 = \overset{-1}{f^2}(\varphi_2)$, $\psi_1(a) = g_i^{-1}(\psi_2(f(a)))$ for all $a \in X_1$ such that $\varphi_1(a) \in X_{1_i}$ (for some $i \in I$). In Theorem 1, it was shown that $\varphi_1 \in \mathrm{End}\, G_1$. Let us show that $\psi_1 \in \mathrm{Hom}(G_1, G_1')$. Let $(a, b) \in \rho_1$, then $(\varphi_1(a), \varphi_1(b)) \in \rho_1$, and elements $\varphi_1(a), \varphi_1(b)$ belong to the same connectivity component X_{1_i} of the graph G_1. As a result, we get the equalities

$$\psi_1(a) = g_i^{-1}(\psi_2(f(a))),\ \psi_1(b) = g_i^{-1}(\psi_2(f(b))).$$

Since f is an isomorphism of G_1 onto G_2, it follows that $(f(a), f(b)) \in \rho_2$. Moreover, $\psi_2 \in \mathrm{Hom}(G_2, G_2')$ implies $(\psi_2(f(a)), \psi_2(f(b))) \in \rho_2'$ and from the fact that g_i is an isomorphism of G_1' onto G_2' it follows that

$$(g_i^{-1}(\psi_2(f(a))), g_i^{-1}(\psi_2(f(b)))) \in \rho_1',$$

i.e. $(\psi_1(a), \psi_1(b)) \in \rho_1'$. Thus $(\varphi_1, \psi_1) \in S_1$, $h(\varphi_1, \psi_1) \in S_2$. Hence, h is surjective and, as a result, bijective.

Let us verify that the mapping h is consistent with the operations of semigroups S_1 and S_2. Let $(\varphi_1, \psi_1), (\varphi_2, \psi_2) \in S_1$. By definition, we have $(\varphi_1, \psi_1) \cdot (\varphi_2, \psi_2) = \varphi_1\varphi_2, \varphi_1\psi_2)$. Then

$$h((\varphi_1, \psi_1) \cdot (\varphi_2, \psi_2)) = h(\varphi_1\varphi_2, \varphi_1\psi_2) = (f^2(\varphi_1\varphi_2), (\varphi_1\psi_2)^{\varphi_1\varphi_2}),$$

$$h(\varphi_1,\psi_1)\cdot h(\varphi_2,\psi_2) = (f^2(\varphi_1),\psi_1^{\varphi_1})\cdot(f^2(\varphi_2),\psi_2^{\varphi_2}) = (f^2(\varphi_1)f^2(\varphi_2), f^2(\varphi_1)\psi_2^{\varphi_2}).$$

Hence $f^2(\varphi_1\varphi_2) = f^2(\varphi_1)f^2(\varphi_2)$.

Consider arbitrary vertex $a \in X_1$ of the graph G_1. We denote $\varphi_1(a) = b$, $\varphi_2(b) = c$, $\psi_2(b) = d$. Then

$$(\varphi_1\varphi_2)(a) = \varphi_2(\varphi_1(a)) = \varphi_2(b) = c, \quad (\varphi_1\psi_2)(a) = \psi_2(\varphi_1(a)) = \psi_2(b) = d.$$

As a result, we get

$$(\varphi_1\psi_2)^{\varphi_1\varphi_2}(f(a)) = g_{\varphi_1\varphi_2(a)}((\varphi_1\psi_2)(a)) = g_c(d),$$
$$f^2(\varphi_1)\psi_2^{\varphi_2}(f(a)) = \psi_2^{\varphi_2}(f^2(\varphi_1)(a)) = \psi_2^{\varphi_2}(f(\varphi_1(a))) = \psi_2^{\varphi_2}(f(b)) = g_{\varphi_2(b)}(\psi_2(b)) = g_c(d).$$

Therefore, right sides are equal and the equality

$$h((\varphi_1,\psi_1)\cdot(\varphi_2,\psi_2)) = h(\varphi_1\varphi_2,\varphi_1\psi_2)$$

holds. Hence, the mapping h is compatible with operations of the semigroups S_1, S_2 and h is an isomorphism of S_1 onto S_2.

Similarly, one can show that the mapping $h : S_1 \to S_2$ is an isomorphism if it is defined by an anti-isomorphism f of the graph G_1 onto the graph G_2 and a family of isomorphisms g_i of the graph G_1' onto the graph G_2', $(i \in I)$. □

The results obtained describe the structure of isomorphisms of the universal graphic automata over quasi-acyclic state graphs and antisymmetric output signal graphs and also establish the relationship between isomorphisms of such automata and isomorphisms of their components (the state graphs, the output signal graphs, the input signal semigroups).

Let G, G' be graphs and $\mathrm{Atm}(G,G')$ be the universal graphic automaton over graphs G, G'. The obtained results on the structure of isomorphisms of the universal graphic automata allow us to study the relationship between the automorphism groups of the automaton $\mathrm{Atm}(G,G')$ and the automorphism groups of its components. We denote by Ant G the set of all anti-automorphisms of the graph G, by $(\mathrm{Aut}\ G)^I$ — the set of families $\{g_i\}_{i\in I}$ of automorphisms of the graph G.

Theorem 4. *Let $G = (X,\rho)$ be a nontrivial quasi-acyclic reflexive graph with connectivity components $\{X_i\}$ $(i \in I)$, $G' = (X',\rho')$ be an antisymmetric reflexive graph, and let $A = \mathrm{Atm}(G,G')$ be the universal graphic automaton with the input signal semigroup $S = \mathrm{End}\ G \times \mathrm{Hom}(G,G')$. Then for the automorphism group $\mathrm{Aut}\ A$ of the automaton A, the automorphism groups $\mathrm{Aut}\ G$, $\mathrm{Aut}\ G'$ of graphs G, G' and the automorphism group $\mathrm{Aut}\ S$ of the input signal semigroup S, the following conditions hold:*

1) $\mathrm{Aut}\ A \cong (\mathrm{Aut}\ G \times \mathrm{Aut}\ G') \cup (\mathrm{Ant}\ G \times \mathrm{Ant}\ G')$;
2) *the automorphism group $\mathrm{Aut}\ S$ is isomorphic to the algebra with the basic set $P = (\mathrm{Aut}\ G \times (\mathrm{Aut}\ G')^I) \cup (\mathrm{Ant}\ G \times (\mathrm{Ant}\ G')^I)$ and the binary operation $\cdot$, which is defined by the formula*

$$(f,\{g_i\}_{i\in I})\cdot(f',\{g_i'\}_{i\in I}) = \left(f\cdot f', \{g_i\cdot g'_{\tilde f(i)}\}_{i\in I}\right), \tag{4}$$

where f, f' are automorphisms (anti-automorphisms) of the graph G, $\{g_i\}_{i\in I}$, $\{g_i'\}_{i\in I}$ are families of automorphisms (anti-automorphisms) of the graph G' and $\tilde f$ is a permutation of the set of indices I induced by the automorphism (anti-automorphism) f.

Proof. The proof of part 1) of the current theorem follows directly from Theorem 2.

Any automorphism (anti-automorphism) f of the graph $G = (X,\rho)$ defines a permutation $\tilde f$ of the set of indices I by the formula $\tilde f(i) = j$, where for any $x \in X_i$, $i \in I$ the condition $f(x) \in X_j$ is satisfied for some $j \in I$.

According to Theorem 3, every automorphism h of the semigroup S is determined by an automorphism (anti-automorphism) f of the graph G and a family of automorphisms (anti-automorphisms) $\{g_i\}_{i\in I}$ of the graph G' so that for all $(\varphi,\psi) \in S$ the formula

$$h(\varphi,\psi) = \left(f^2(\varphi),\psi^{\varphi}\right)$$

holds, where $\psi^{\varphi}(f(a)) = g_i(\psi(a))$ for all $a \in X$ such that the condition $\varphi(a) \in X_i$ is satisfied for some $i \in I$. This implies that the formula $\Gamma(h) = (f, \{g_i\}_{i\in I})$ ($h \in \mathrm{Aut}\ S$) defines the bijection $\Gamma : \mathrm{Aut}\ S \to P$. Let us show that for all $h_1, h_2 \in \mathrm{Aut}\ S$ the condition $\Gamma(h_1 \cdot h_2) = \Gamma(h_1) \cdot \Gamma(h_2)$ is satisfied. Let $\Gamma(h_1) = (f_1, \{g_i^1\}_{i\in I})$, $\Gamma(h_2) = (f_2, \{g_i^2\}_{i\in I})$, where $f_1, f_2 \in \mathrm{Aut}\ G$ (or $\mathrm{Ant}\ G$), $\{g_i^1\}_{i\in I}$, $\{g_i^2\}_{i\in I} \in (\mathrm{Aut}\ G')^I$ (or $(\mathrm{Ant}\ G')^I$). By the definition of the binary operation $\cdot$ in the algebra P the following equalities hold:

$$\Gamma(h_1)\cdot\Gamma(h_2) = \left(f_1, \{g_i^1\}_{i\in I}\right)\cdot\left(f_2, \{g_i^2\}_{i\in I}\right) = \left(f_1\cdot f_2, \{g_i^1\cdot g^2_{\tilde f_1(i)}\}_{i\in I}\right).$$

Let us denote $h_1 \cdot h_2 = h$, $f_1 \cdot f_2 = f$ and $g_i^1 \cdot g^2_{\tilde f_1(i)} = g_i$ for every $i \in I$. Let $(\varphi_1, \psi_1) \in S$ and $h_1(\varphi_1, \psi_1) = (\varphi_2, \psi_2)$. On the other hand, by Theorem 3 we have $h_1(\varphi_1, \psi_1) = \left(f_1^2(\varphi_1), \psi_1^{\varphi_1}\right)$, where $\psi_1^{\varphi_1}(a) = g_i^1\left(\psi_1\left(f_1^{-1}(a)\right)\right)$ for any $a \in X$ such that the condition $\varphi_1\left(f_1^{-1}(a)\right) \in X_i$ holds for some $i \in I$. Hence $\varphi_2 = f_1^2(\varphi_1)$, $\psi_2 = \psi_1^{\varphi_1}$. Similarly, for $(\varphi_2, \psi_2) \in S$ we get $h_2(\varphi_2, \psi_2) = \left(f_2^2(\varphi_2), \psi_2^{\varphi_2}\right)$, where $\psi_2^{\varphi_2}(a) = g_i^2\left(\psi_2\left(f_2^{-1}(a)\right)\right)$ for any $a \in X$ such that $\varphi_2\left(f_2^{-1}(a)\right) \in X_i$ holds for some $i \in I$.

As a result, we get

$$f_2^2(\varphi_2) = f_2^2\left(f_1^2\left(\varphi_1\right)\right) = (f_1f_2)^2(\varphi_1) = f^2(\varphi_1).$$

In addition, for every $a \in X$ we get

$$\varphi_2\left(f_2^{-1}(a)\right) = \left(f_1^2(\varphi_1)\right)\left(f_2^{-1}(a)\right) = \left(f_1^{-1}\varphi_1 f_1\right)\left(f_2^{-1}(a)\right) =$$
$$= \left(f_2^{-1}f_1^{-1}\varphi_1 f_1\right)(a) = \left(\left(f_1f_2\right)^{-1}\varphi_1 f_1\right)(a) = f_1\left(\varphi_1\left(\left(f_1f_2\right)^{-1}(a)\right)\right) = f_1\left(\varphi_1\left(f^{-1}(a)\right)\right)$$

and

$$\psi_2\left(f_2^{-1}(a)\right) = \psi_1^{\varphi_1}\left(f_2^{-1}(a)\right) = g_i^1\left(\psi_1\left(f_1^{-1}\left(f_2^{-1}(a)\right)\right)\right) =$$
$$= g_i^1\left(\left(f_2^{-1}f_1^{-1}\psi_1\right)(a)\right) = g_i^1\left(\left(\left(f_1f_2\right)^{-1}\psi_1\right)(a)\right) = g_i^1\left(\left(f^{-1}\psi_1\right)(a)\right) = g_i^1\left(\psi_1\left(f^{-1}(a)\right)\right),$$

where the index $i \in I$ is such that

$$\varphi_1\left(f_1^{-1}\left(f_2^{-1}(a)\right)\right) = \left(f_2^{-1}f_1^{-1}\varphi_1\right)(a) = \left(\left(f_1f_2\right)^{-1}\varphi_1\right)(a) = \left(f^{-1}\varphi_1\right)(a) = \varphi_1\left(f^{-1}(a)\right) \in X_i.$$

Consequently,

$$h\left(\varphi_1, \psi_1\right) = \left(h_1h_2\right)\left(\varphi_1, \psi_1\right) = h_2\left(h_1\left(\varphi_1, \psi_1\right)\right) = h_2\left(\varphi_2, \psi_2\right) = \left(f_2^2(\varphi_2), \psi_2^{\varphi_2}\right) = \left(f^2\left(\varphi_1\right), \psi_2^{\varphi_2}\right),$$

where for every $a \in X$ equations hold

$$\psi_2^{\varphi_2}(a) = g^2_{\tilde f_1(i)}\left(\psi_2\left(f_2^{-1}(a)\right)\right) = g^2_{\tilde f_1(i)}\left(g_i^1\left(\psi_1\left(f^{-1}(a)\right)\right)\right) =$$
$$= \left(g_i^1 g^2_{\tilde f_1(i)}\right)\left(\psi_1\left(f^{-1}(a)\right)\right) = g_i\left(\psi_1\left(f^{-1}(a)\right)\right),$$

since the vertex $\varphi_1\left(f^{-1}(a)\right)$ belongs to the connectivity component X_i, the vertex $\varphi_2\left(f_2^{-1}(a)\right) = f_1\left(\varphi_1\left(f^{-1}(a)\right)\right)$ belongs to the connectivity component $X_{\tilde f(i)}$ and $g_i^1 g^2_{\tilde f_1(i)} = g_i$.

It is easy to see that $f = f_1f_2$ is an automorphism (anti-automorphism) of the graph G, for each index $i \in I$ the mapping g_i is an automorphism (anti-automorphism) of the graph G'. Hence $\Gamma(h_1 \cdot h_2) = (f, \{g_i\}_{i\in I})$, where $f = f_1f_2$, $g_i = g_i^1 g^2_{\tilde f_1(i)}$ for all $i \in I$. It follows that

$$\Gamma(h_1)\cdot\Gamma(h_2) = \left(f_1f_2, \{g_i^1 g^2_{\tilde f_1(i)}\}_{i\in I}\right) = \Gamma(h_1\cdot h_2)$$

and the mapping $\Gamma : \mathrm{Aut}\ S \to P$ is an isomorphism. □

Corollary 1. *Let* $G = (X, \rho)$ *be a nontrivial quasi-acyclic reflexive graph,* $G' = (X', \rho')$ *be an antisymmetric reflexive graph, and let* $A = \mathrm{Atm}(G, G')$ *be the universal graphic automaton with the input signal semigroup* $S = \mathrm{End}\ G \times \mathrm{Hom}(G, G')$*. Then the automorphism group* $\mathrm{Aut}\ S$ *of the input signal semigroup* S *is isomorphic to a subgroup of the union of the wreath products [14] of the groups*

$$\left((G', \mathrm{Aut}\ G') \wr (G, \mathrm{Aut}\ G)\right) \cup \left((G', \mathrm{Ant}\ G') \wr (G, \mathrm{Ant}\ G)\right), \tag{5}$$

which consists of ordered pairs (ψ, φ)*, where* $\varphi \in \mathrm{Aut}\ G$ *and* $\psi \in (\mathrm{Aut}\ G')^G$ *or* $\varphi \in \mathrm{Ant}\ G$ *and* $\psi \in (\mathrm{Ant}\ G')^G$*,* $\psi(a) = \psi(b)$ *for all adjacent vertexes* $a, b \in X$ *of the graph* G*.*

Corollary 2. *Let* $G = (X, \rho)$ *be a nontrivial quasi-acyclic reflexive connected graph,* $G' = (X', \rho')$ *be an antisymmetric reflexive graph, and let* $A = \mathrm{Atm}(G, G')$ *be a universal graphic automaton with the input signal semigroup* $S = \mathrm{End}\ G \times \mathrm{Hom}(G, G')$*. Then, the automorphism group* $\mathrm{Aut}\ S$ *of the input signal semigroup* S *is isomorphic to the group* $(\mathrm{Aut}\ G \times \mathrm{Aut}\ G') \cup (\mathrm{Ant}\ G \times \mathrm{Ant}\ G')$.

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