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Article

On a time-optimal control problem for a heat conduction equation with involution

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Abstract. In this paper, we consider a boundary control problem for a heat conduction equation with involution in a bounded one-dimensional domain. The solution with the control function on the border of the rod is given. The constraints on the control are determined to ensure that the average value of the solution within the considered domain attains a given value. The considered control problem is reduced to the Volterra integral equation, which is the first type, using the Fourier method. The proof of the existence of admissible control is related to the existence of a solution of the integral equation. The existence of the control function was proved by the Laplace transform method, and the estimate of the minimum time to reach the given average temperature in the rod was found.
Keywords: initial-boundary problem, heat equation, minimal time, integral equation, admissible control, Laplace transform, involution
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О задаче оптимального по времени управления для уравнения теплопроводности с инволюцией

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Аннотация. В данной работе рассматривается задача граничного управления для уравнения теплопроводности с инволюцией в ограниченной одномерной области. Приводится решение с функцией управления на границе стержня. Определены ограничения на управление, обеспечивающие достижение средним значением решения в рассматриваемой области заданного значения. Рассматриваемая задача управления сводится к интегральному уравнению Вольтерра, которое является первым типом, с помощью метода Фурье. Доказательство существования допустимого управления связано с существованием решения интегрального уравнения. Методом преобразования Лапласа доказано

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существование функции управления и найдена оценка минимального времени достижения заданной средней температуры в стержне.

Ключевые слова: начально-краевая задача, уравнение теплопроводности, минимальное время, интегральное уравнение, допустимое управление, преобразование Лапласа, инволюция

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Introduction

It is known that due to the widespread use of partial differential equations in physics and engineering, there is always a great interest in the study of boundary control problems. Therefore, in recent years, the control problems for heat conduction equations have been widely studied by many researchers.

The optimal control problem for the parabolic type equations was studied by Fattorini and Friedman [1, 2]. Control problems for the infinite-dimensional case were studied by Egorov [3], who generalized Pontryagin's maximum principle to a class of equations in Banach space, and the proof of a bang-bang principle was shown under the particular conditions. The time-varying bang–bang property of time optimal controls for the heat equation and its applications is studied in [4].

The boundary control problem for a heat equation with a piecewise smooth boundary in an n-dimensional domain was studied in [5], and an estimate for the minimum time required to reach a given average temperature was found. In [6], the control problem for the heat conduction equation with the Robin boundary condition is studied, and a mathematical model of the heating process of a cylindrical domain is developed. Control problems for the heat transfer equation in the three-dimensional domain are studied in [7].

The control problem related to the inhomogeneous heat transfer equation was studied in [8], and the existence of the admissible control with the additional condition of the weight function was proved by an integral constraint. Control problems for heat equations in bounded one and two-dimensional domains are studied in works [9–11]. In these articles, an estimated was found for the minimum time required to heat a bounded domain to an estimated average temperature. The existence of a control function is proved by the Laplace transform method.

Basic information on optimal control problems is given in detail in monographs by Lions and Fursikov [12, 13]. General numerical optimization and optimal control for second-order parabolic equations have been studied in many publications, such as [14]. In [15], some practical problems for control problems related to heat equations are studied.

It is known that in recent years, due to the increasing interest in physics and mathematics, the boundary problems related to heat diffusion equations involving involution have been widely studied. In [16], a boundary value problem for the heat equation associated with involution in a one-dimensional domain is studied. Many boundary value problems for parabolic-type equations with involution were studied in works [17, 18].

In [19], the control problem associated with a pseudo-parabolic type equation in a one-dimensional domain was studied, and the existence of an admissible control was proved using the Laplace transform method. The boundary control problem in a bounded two-dimensional domain for a pseudo-parabolic type equation was studied in [20].

In this work, the boundary control problem for the heat equation with involution is considered. The main control problem in this work is presented in Section 1. The boundary control problem studied in this work is reduced to the Volterra integral equation of the first kind by the Fourier method (Section 2). In Section 3, the existence of a solution to the integral equation is proved using the Laplace transform method. Section 4 gives an estimate of the minimum time required to reach a given average temperature of the rod.

1. Statement of problem

In this paper, we consider the heat equation with involution in the domain $\Omega_T := (0,\pi)\times(0,\infty)$

$$u_t(x,t) - u_{xx}(x,t) + \alpha u_{xx}(\pi - x, t) = 0, \quad (x,t) \in \Omega_T, \tag{1}$$

with Dirichlet boundary conditions

$$u(0,t) = \nu(t), \quad u(\pi,t) = 0, \quad t \geqslant 0, \tag{2}$$

and initial condition

$$u(x,0) = 0, \quad 0 \leqslant x \leqslant \pi, \tag{3}$$

where α is a nonzero real number such that $|\alpha| < 1$ and $\nu(t)$ is the control function, which gives the flow amplitude.

Let $M > 0$ be some given constant. We say that the control function $\nu(t)$ is admissible if it is continuously differentiable on the half-line $t \geqslant 0$ and satisfies the conditions

$$\nu(0) = 0, \quad |\nu(t)| \leqslant M, \quad t \geqslant 0.$$

Differential equations with modified arguments are equations in which the unknown function and its derivatives are evaluated with modifications of time or space variables; such equations are called, in general, functional differential equations. Among such equations, one can single out equations with involutions [21]. It is known that a function $g(x) \not\equiv x$ maps bijectively a set of real numbers D, such that $g(g(x)) = x$ or $g^{-1}(x) = g(x)$ is called an involution on D (see [22, 23]).

Assume that the *weight function* $\rho(x) \in W_2^1([0,\pi])$ satisfies the conditions

$$\rho(x) \geqslant 0, \quad \rho'(x) \leqslant 0, \quad \int\limits_0^\pi \rho(x)\, dx = 1, \quad 0 \leqslant x \leqslant \pi. \tag{4}$$

Time-optimal problem. *Let $\theta > 0$ be a given constant. Problem consists looking for the minimal value of $T > 0$ so that for $t > 0$ the solution $u(x,t)$ of the problem (1)–(3) with control function $\nu(t)$ exists and for some $T_1 > T$ satisfies the equation*

$$\int\limits_0^\pi \rho(x) u(x,t)\, dx = \theta, \quad T \leqslant t \leqslant T_1. \tag{5}$$

The physical meaning of equation (5) is the average temperature in the rod. Our main goal in this work is to find the minimum time estimate for the average temperature in the rod to be equal to θ.

Remark 1. It is known that boundary control problems for the non-homogeneous heat equation in the case $\alpha = 0$ are studied in detail in works [24].

We consider the spectral problem

$$X''(x) - \alpha X''(\pi - x) + \lambda X(x) = 0, \quad 0 < x < \pi,$$
$$X(0) = X(\pi) = 0, \quad 0 \leqslant x \leqslant \pi,$$

where $|\alpha| < 1$, $\alpha \in \mathbb{R}\setminus\{0\}$. It is proved in [17,18] that expressing the solution of spectral problem in terms of the sum of even and odd functions, one finds the following eigenvalues:

$$\lambda_{2k} = 4(1+\alpha)k^2, \quad k \in \mathbb{N}, \tag{6}$$

$$\lambda_{2k+1} = (1-\alpha)(2k+1)^2, \quad k \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}, \tag{7}$$

and eigenfunctions

$$X_{2k} = \sin 2kx, \quad k \in \mathbb{N}, \quad X_{2k+1} = \sin(2k+1)x, \quad k \in \mathbb{N}_0.$$

Let

$$\rho(x) = \sum_{k=1}^{\infty} \rho_k \sin kx, \quad x \in (0,\pi), \tag{8}$$

where

$$\rho_k = \frac{2}{\pi}\int\limits_0^\pi \rho(x)\sin kx\, dx, \quad k = 1, 2, \cdots . \tag{9}$$

Denote

$$\beta_{2k+1} = (1-\alpha)(2k+1)\rho_{2k+1}, \quad k \in \mathbb{N}_0, \quad \beta_{2k} = (1+\alpha)2k\rho_{2k}, \quad k \in \mathbb{N}. \tag{10}$$

Theorem 1. *Let*

$$0 < \theta < \frac{\beta_1 M}{\lambda_1}.$$

Set

$$T^* = -\frac{1}{\lambda_1}\ln\left(1 - \frac{\theta\lambda_1}{\beta_1 M}\right).$$

Then a solution T_{min} of the Time-Optimal Problem exists and the estimate $T_{min} \leqslant T^*$ is valid.

We will consider the proof of Theorem 1 step by step in the next sections.

2. Integral equation for control function

In this section, we consider how the given control problem can be reduced to a Volterra integral equation of the first kind.

By the solution of the problem (1)–(3) we mean function $u(x,t)$, expressed the form

$$u(x,t) = \nu(t)\frac{\pi - x}{\pi} - w(x,t), \tag{11}$$

where the function $w(x,t) \in C^{2,1}_{x,t}(\Omega_T)\cap C(\bar{\Omega}_T)$ is the solution to the problem

$$w_t(x,t) - w_{xx}(x,t) + \alpha w_{xx}(\pi - x, t) = \frac{\pi - x}{\pi}\nu'(t),$$

with initial-boundary conditions

$$w(0,t) = w(\pi,t) = 0, \quad w(x,0) = 0.$$

Thus, we have (see [25])

$$w(x,t) = \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}\left(\int\limits_0^t e^{-\lambda_{2k+1}(t-s)}\nu'(s)\,ds\right)\sin(2k+1)x +$$

$$+\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\Big(\int\limits_0^t e^{-\lambda_{2k}(t-s)}\nu'(s)\,ds\Big)\sin 2kx. \tag{12}$$

It follows from (11) and (12), we get the solution of the problem (1)–(3):

$$u(x,t)=\frac{\pi-x}{\pi}\nu(t)-\frac{2}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}\Big(\int\limits_0^t e^{-\lambda_{2k+1}(t-s)}\nu'(s)\,ds\Big)\sin(2k+1)x-$$

$$-\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\Big(\int\limits_0^t e^{-\lambda_{2k}(t-s)}\nu'(s)\,ds\Big)\sin 2kx. \tag{13}$$

From (13) and the condition (5), we can write

$$f(t)=\int\limits_0^\pi \rho(x)u(x,t)\,dx=\nu(t)\int\limits_0^\pi \rho(x)\frac{\pi-x}{\pi}\,dx-$$

$$-\frac{2}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}\int\limits_0^t e^{-\lambda_{2k+1}(t-s)}\nu'(s)\,ds\int\limits_0^\pi \rho(x)\sin(2k+1)x\,dx-$$

$$-\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\int\limits_0^t e^{-\lambda_{2k}(t-s)}\nu'(s)\,ds\int\limits_0^\pi \rho(x)\sin 2kx\,dx.$$

where $f(t)=\theta$ for $T\leqslant t\leqslant T_1$.

Then from (8), we have

$$f(t)=\ \nu(t)\int\limits_0^\pi \rho(x)\frac{\pi-x}{\pi}\,dx-\sum_{k=0}^{\infty}\frac{\rho_{2k+1}}{2k+1}\int\limits_0^t e^{-\lambda_{2k+1}(t-s)}\nu'(s)\,ds-$$

$$-\sum_{k=1}^{\infty}\frac{\rho_{2k}}{2k}\int\limits_0^t e^{-\lambda_{2k}(t-s)}\nu'(s)\,ds. \tag{14}$$

By the condition $\nu(0)=0$ and (14) we may write

$$f(t)=\nu(t)\int\limits_0^l \rho(x)\frac{l-x}{l}\,dx-\nu(t)\sum_{k=1}^{\infty}\frac{\rho_k}{k}+(1-\alpha)\sum_{k=0}^{\infty}(2k+1)\rho_{2k+1}\int\limits_0^t e^{-\lambda_{2k+1}(t-s)}\nu(s)\,ds+$$

$$+(1+\alpha)\sum_{k=1}^{\infty}2k\rho_{2k}\int\limits_0^t e^{-\lambda_{2k}(t-s)}\nu(s)\,ds.$$

According to Parseval equality, we get

$$\int\limits_0^\pi \rho(x)\frac{\pi-x}{\pi}\,dx=\sum_{k=1}^{\infty}\frac{\rho_k}{k}.$$

As a result we can write

$$f(t)=(1-\alpha)\sum_{k=0}^{\infty}(2k+1)\rho_{2k+1}\int\limits_0^t e^{-\lambda_{2k+1}(t-s)}\nu(s)\,ds*+$$

$$+(1+\alpha)\sum_{k=1}^{\infty} 2k\rho_{2k}\int\limits_0^t e^{-\lambda_{2k}(t-s)}\nu(s)\,ds. \tag{15}$$

Let us introduce the function

$$K(t)=\sum_{k=0}^{\infty}\beta_{2k+1}e^{-\lambda_{2k+1}t}+\sum_{k=1}^{\infty}\beta_{2k}e^{-\lambda_{2k}t}, \quad t>0, \tag{16}$$

where β_{2k+1} and β_{2k} are defined by (10).

Then equality (15) takes the form

$$\int\limits_0^t K(t-s)\nu(s)ds=f(t), \quad t>0, \tag{17}$$

where $f(t)=\theta=\mathrm{const}>0$ for $T\leqslant t\leqslant T_1$.

The resulting Volterra integral equation (17) is the main equation for admissible control $\nu(t)$.

For any $M_0>0$, we denote $W(M_0)$ the set of function $f\in W_2^2(-\infty,+\infty)$, $f(t)=0$ for $t\leqslant 0$ which satisfying the condition

$$\|f\|_{W_2^2(R_+)}\leqslant M_0.$$

Theorem 2. *There exists $M_0>0$ such that for any function $f\in W(M_0)$ the solution $\nu(t)$ of the equation (17) exists and satisfies the condition $|\nu(t)|\leqslant M$.*

Lemma 1 ([26]). *Let $\psi(x)\geqslant 0$ and $\psi'(x)\leqslant 0$ on $x\in[0,\infty)$. Then the following inequality is valid:*

$$\int\limits_0^{n\pi}\psi(x)\sin x\,dx\geqslant 0, \quad n=1,2,\ldots.$$

Lemma 2. *For the coefficients $\{\rho_k\}_{k\in N}$ defined by (9), the following estimate is valid:*

$$0\leqslant\rho_k\leqslant\frac{C}{k}, \quad k=1,2,\ldots,$$

where $C=\mathrm{const}>0$.

Proof. According to Lemma 1, we have

$$\rho_k\geqslant 0, \quad k=1,2,\ldots.$$

From (9), we write

$$\rho_k=\frac{2}{\pi}\int\limits_0^{\pi}\rho(x)\sin kx\,dx=-\frac{2}{\pi}\,\rho(x)\frac{1}{k}\cos kx\Big|_{x=0}^{x=\pi}+\frac{2}{k\pi}\int\limits_0^{\pi}\rho'(x)\cos kx\,dx=$$
$$=\frac{2}{k\pi}\Big(\rho(0)-(-1)^k\rho(\pi)\Big)+\frac{o(1)}{k}.$$

It is clear that

$$\rho(0)-(-1)^k\rho(\pi)\geqslant 0, \quad k=1,2,\ldots,$$

where function $\rho(x)$ is satisfying conditions (4).

Then we obtain $0\leqslant\rho_k\leqslant\frac{C}{k}$. □

Lemma 3. *Let $|\alpha| < 1$. Then the following estimate is valid:*

$$0 < K(t) \leqslant \frac{C_\alpha}{\sqrt{t}}, \quad 0 < t \leqslant 1,$$

where function $K(t)$ is defined by (16) and C_α is a constant only depending on α.

Proof. For any $p > 0$, consider the following relations:

$$\sum_{n=1}^{\infty} e^{-pn^2} = \sum_{n=1}^{\infty} \int_{n}^{n+1} e^{-p[s]^2}\, ds = \int_{1}^{\infty} e^{-p[s]^2}\, ds = \int_{1}^{\infty} e^{-ps^2} e^{p(s^2-[s]^2)}\, ds,$$

where $[s]$ is the integer part of s.

Note that $e^{p(s^2-[s]^2)} = e^{p(s-[s])(s+[s])} \leqslant e^{2ps}$. Then we obtain

$$\int_{1}^{\infty} e^{-ps^2} e^{p(s^2-[s]^2)}\, ds \leqslant \int_{1}^{\infty} e^{-ps^2+2ps}\, ds = e^{p} \int_{1}^{\infty} e^{-p(s-1)^2}\, ds.$$

Hence, for $0 < p \leqslant \text{const}$ we get

$$\sum_{n=1}^{\infty} e^{-pn^2} \leqslant \int_{1}^{\infty} e^{-ps^2} e^{p(s^2-[s]^2)}\, ds \leqslant e^{p} \int_{0}^{\infty} e^{-ps^2} ds \leqslant \frac{C}{\sqrt{p}}. \tag{18}$$

From (6), (7) and (18), we have

$$\sum_{k=0}^{\infty} e^{-\lambda_{2k+1}^2 t} \leqslant \frac{1}{\sqrt{1-\alpha}} \frac{C}{\sqrt{t}},$$

and

$$\sum_{k=1}^{\infty} e^{-\lambda_{2k}^2 t} \leqslant \frac{1}{\sqrt{1+\alpha}} \frac{C}{\sqrt{t}}.$$

From (10) and Lemma 2, we may write

$$0 \leqslant \beta_{2k} \leqslant C(1+\alpha), \quad k \in \mathbb{N},$$

and

$$0 \leqslant \beta_{2k+1} \leqslant C(1-\alpha), \quad k \in \mathbb{N}_0.$$

Consequently, we get the following estimate for the function $K(t)$:

$$0 < K(t) \leqslant C_1 \frac{\sqrt{1-\alpha}}{\sqrt{t}} + C_2 \frac{\sqrt{1+\alpha}}{\sqrt{t}} \leqslant \frac{C_\alpha}{\sqrt{t}},$$

where $C_\alpha = \max\{C_1\sqrt{1-\alpha},\, C_2\sqrt{1+\alpha}\}$. □

3. Proof of Theorem 2

We use the Laplace transform method to solve equation (17). We introduce the notation

$$\widetilde{\nu}(p) = \int_{0}^{\infty} e^{-pt} \nu(t)\, dt.$$

Then we use the Laplace transform to obtain the following equation

$$\widetilde{f}(p) = \int\limits_0^\infty e^{-pt}\,dt \int\limits_0^t K(t-s)\nu(s)\,ds = \widetilde{K}(p)\widetilde{\nu}(p).$$

Consequently, we obtain

$$\widetilde{\nu}(p) = \frac{\widetilde{f}(p)}{\widetilde{K}(p)}, \quad \text{where} \quad p = \sigma + i\tau, \quad \sigma > 0, \quad \tau \in \mathbb{R},$$

and

$$\nu(t) = \frac{1}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty} \frac{\widetilde{f}(p)}{\widetilde{K}(p)} e^{pt}dp = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} \frac{\widetilde{f}(\sigma+i\tau)}{\widetilde{K}(\sigma+i\tau)} e^{(\sigma+i\tau)t}d\tau. \tag{19}$$

Lemma 4. *The following estimate*

$$|\widetilde{K}(\sigma+i\tau)| \geqslant \frac{C_\sigma}{\sqrt{1+\tau^2}}, \quad \sigma > 0, \quad \tau \in \mathbb{R},$$

is valid, where $C_\sigma > 0$ is a constant only depending on σ.

Proof. Using the Laplace transform, we can write

$$\widetilde{K}(p) = \int\limits_0^\infty K(t)e^{-pt}\,dt = \sum_{k=0}^\infty \beta_{2k+1}\int\limits_0^\infty e^{-(p+\lambda_{2k+1})t}\,dt + \sum_{k=1}^\infty \beta_{2k}\int\limits_0^\infty e^{-(p+\lambda_{2k})t}\,dt =$$
$$= \sum_{k=0}^\infty \frac{\beta_{2k+1}}{p+\lambda_{2k+1}} + \sum_{k=1}^\infty \frac{\beta_{2k}}{p+\lambda_{2k}},$$

where $K(t)$ is defined by (16) and

$$\widetilde{K}(\sigma+i\tau) = \sum_{k=0}^\infty \frac{\beta_{2k+1}}{\sigma+\lambda_{2k+1}+i\tau} + \sum_{k=1}^\infty \frac{\beta_{2k}}{\sigma+\lambda_{2k}+i\tau} =$$
$$= \sum_{k=0}^\infty \frac{\beta_{2k+1}(\sigma+\lambda_{2k+1})}{(\sigma+\lambda_{2k+1})^2+\tau^2} + \sum_{k=1}^\infty \frac{\beta_{2k}(\sigma+\lambda_{2k})}{(\sigma+\lambda_{2k})^2+\tau^2} - i\tau\sum_{k=0}^\infty \frac{\beta_{2k+1}}{(\sigma+\lambda_{2k+1})^2+\tau^2} -$$
$$-i\tau\sum_{k=1}^\infty \frac{\beta_{2k}}{(\sigma+\lambda_{2k})^2+\tau^2} = \mathrm{Re}\widetilde{K}(\sigma+i\tau) + i\mathrm{Im}\widetilde{K}(\sigma+i\tau),$$

where

$$\mathrm{Re}\widetilde{K}(\sigma+i\tau) = \sum_{k=0}^\infty \frac{\beta_{2k+1}(\sigma+\lambda_{2k+1})}{(\sigma+\lambda_{2k+1})^2+\tau^2} + \sum_{k=1}^\infty \frac{\beta_{2k}(\sigma+\lambda_{2k})}{(\sigma+\lambda_{2k})^2+\tau^2},$$
$$\mathrm{Im}\widetilde{K}(\sigma+i\tau) = -\tau\sum_{k=0}^\infty \frac{\beta_{2k+1}}{(\sigma+\lambda_{2k+1})^2+\tau^2} - \tau\sum_{k=1}^\infty \frac{\beta_{2k}}{(\sigma+\lambda_{2k})^2+\tau^2}.$$

We know that

$$(\sigma+\lambda_k)^2+\tau^2 \leqslant [(\sigma+\lambda_k)^2+1](1+\tau^2),$$

and we get

$$\frac{1}{(\sigma+\lambda_k)^2+\tau^2} \geqslant \frac{1}{1+\tau^2}\frac{1}{(\sigma+\lambda_k)^2+1}. \tag{20}$$

Thus, according to (20) we can obtain the estimates

$$|\mathrm{Re}\widetilde{K}(\sigma+i\tau)| = \sum_{k=0}^{\infty}\frac{\beta_{2k+1}(\sigma+\lambda_{2k+1})}{(\sigma+\lambda_{2k+1})^2+\tau^2} + \sum_{k=1}^{\infty}\frac{\beta_{2k}(\sigma+\lambda_{2k})}{(\sigma+\lambda_{2k})^2+\tau^2} \geqslant$$

$$\geqslant \frac{1}{1+\tau^2}\sum_{k=0}^{\infty}\frac{\beta_{2k+1}(\sigma+\lambda_{2k+1})}{(\sigma+\lambda_{2k+1})^2+1} = \frac{C_{1,\sigma}}{1+\tau^2}, \tag{21}$$

and

$$|\mathrm{Im}\widetilde{K}(\sigma+i\tau)| \geqslant |\tau|\sum_{k=0}^{\infty}\frac{\beta_{2k+1}}{(\sigma+\lambda_{2k+1})^2+\tau^2} \geqslant \frac{|\tau|}{1+\tau^2}\sum_{k=0}^{\infty}\frac{\beta_{2k+1}}{(\sigma+\lambda_{2k+1})^2+1} = \frac{C_{2,\sigma}|\tau|}{1+\tau^2}, \tag{22}$$

where $C_{1,\sigma}$, $C_{2,\sigma}$ as follows

$$C_{1,\sigma} = \sum_{k=0}^{\infty}\frac{\beta_{2k+1}(\sigma+\lambda_{2k+1})}{(\sigma+\lambda_{2k+1})^2+1}, \quad C_{2,\sigma} = \sum_{k=0}^{\infty}\frac{\beta_{2k+1}}{(\sigma+\lambda_{2k+1})^2+1}.$$

From (21) and (22), we have the estimate

$$|\widetilde{K}(\sigma+i\tau)|^2 = |\mathrm{Re}\widetilde{K}(\sigma+i\tau)|^2 + |\mathrm{Im}\widetilde{K}(\sigma+i\tau)|^2 \geqslant \frac{\min(C_{1,\sigma}^2, C_{2,\sigma}^2)}{1+\tau^2},$$

and

$$|\widetilde{K}(\sigma+i\tau)| \geqslant \frac{C_\sigma}{\sqrt{1+\tau^2}}, \quad \text{where} \quad C_\sigma = \min(C_{1,\sigma}, C_{2,\sigma}).$$

□

Then proceed to the limit as $\sigma \to 0$ from (19), we obtain

$$\nu(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\widetilde{f}(i\tau)}{\widetilde{K}(i\tau)}e^{i\tau t}d\tau. \tag{23}$$

Lemma 5 ([24]). *Let $f(t) \in W(M_0)$. Then, for the imaginary part of the Laplace transform of function $f(t)$, the inequality*

$$\int_{-\infty}^{+\infty}|\widetilde{f}(i\tau)|\sqrt{1+\tau^2}d\tau \leqslant C_1\|f\|_{W_2^2(R_+)},$$

is valid, where $C_1 > 0$ is a constant.

Proof of Theorem 2. From (23) and Lemmas 4 and 5, we can write

$$|\nu(t)| \leqslant \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{|\widetilde{f}(i\tau)|}{|\widetilde{K}(i\tau)|}d\tau \leqslant \frac{1}{2\pi C_0}\int_{-\infty}^{+\infty}|\widetilde{f}(i\tau)|\sqrt{1+\tau^2}d\tau \leqslant \frac{C_1}{2\pi C_0}\|f\|_{W_2^2(R_+)} \leqslant \frac{C_1 M_0}{2\pi C_0} = M,$$

where

$$M_0 = \frac{2\pi C_0}{C_1}M.$$

4. Proof of Theorem 1

We consider the integral equation

$$\int\limits_0^t K(t-s)\nu(s)\,ds = \theta, \quad T \leqslant t \leqslant T_1,$$

where $K(t)$ is defined by (16).

Lemma 6. *The following estimate is valid:*

$$K(t) \geqslant \beta_1 e^{-\lambda_1 t},$$

where the function $K(t)$ is defined by Eq. (16).

The proof of his proposition is based on the fact that the functional series defined by (16) is non-negative.

We introduce a specific heating as

$$H(t) = \int\limits_0^t K(t-s)\,ds = \int\limits_0^t K(s)\,ds.$$

The physical meaning of this function is the average temperature in the rod (see [5]). It is known $H(0) = 0$ and $H'(t) = K(t) > 0$.

We set

$$H^* = \lim_{t\to\infty} H(t) = \int\limits_0^\infty K(s)\,ds.$$

Certainly, the average temperature of the rod in the case where the heater is acting with unit load cannot exceed H^*.

It is clear that H^* is finite. Indeed, from (10) and (16) we have

$$H^* = \sum_{k=0}^\infty \frac{\beta_{2k+1}}{\lambda_{2k+1}} + \sum_{k=1}^\infty \frac{\beta_{2k}}{\lambda_{2k}} = \sum_{k=0}^\infty \frac{\rho_{2k+1}}{2k+1} + \sum_{k=1}^\infty \frac{\rho_{2k}}{2k} = \sum_{k=1}^\infty \frac{\rho_k}{k} < \infty.$$

Lemma 7 ([24])**.** *Let $0 < \theta < MH^*$. Then there exist $T > 0$ and a real-valued measurable function $\nu(t)$, and the following equality*

$$\int\limits_0^T K(T-s)\nu(s)\,ds = \theta,$$

is valid.

It is clear that the value T, which was found in Proposition 6, gives a solution to the problem. Namely, T is the root of the equation

$$H(T) = \frac{\theta}{M}. \tag{24}$$

Lemma 8. *Let*

$$0 < \theta < \frac{\beta_1 M}{\lambda_1}.$$

Then there exists $T > 0$ so that

$$T < -\frac{1}{\lambda_1}\ln\left(1 - \frac{\theta\lambda_1}{\beta_1 M}\right),$$

and Eq. (24) is fulfilled.

Proof. For obtaining the required estimate, we use Lemma 6. We may write

$$H(t) = \int_0^t K(s)\,ds \geqslant \beta_1 \int_0^t e^{-\lambda_1 s} ds = \frac{\beta_1}{\lambda_1}\Big(1 - e^{-\lambda_1 t}\Big). \tag{25}$$

Consider the following equation for defining T^*:

$$\frac{\beta_1}{\lambda_1}\Big(1 - e^{-\lambda_1 T^*}\Big) = \frac{\theta}{M}. \tag{26}$$

Then

$$T^* = -\frac{1}{\lambda_1}\ln\Big(1 - \frac{\theta\lambda_1}{\beta_1 M}\Big).$$

In accordance with (25) and (26), we have

$$0 < \frac{\theta}{M} \leqslant H(T^*).$$

Then obviously there exists T $(0 < T < T^*)$, which is a solution to the equation (24). $\square$

The proof of Theorem 1 follows from Lemma 8.

Conclusion

Note that in the case where the temperature θ is small enough, the value of T^* can be replaced by the following one:

$$T^* \cong \frac{\theta}{\beta_1 M}.$$

Hence, in this case, the estimate of optimal time given by Theorem 1 is proportional to required temperature θ and inversely proportional to the size of the rod l and to the maximum output of the heat source M.

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